Pattern Classes of Permutations: Constructions, Atomicity and the Finite Basis Property

Robert Brignall

September 21, 2004

Contents

Acknowledgements

I should like to thank Dr Nik Ruškuc for his suggestions, ideas and supervision of this dissertation, and for tolerating my frequent absences from the department over the summer. I should also like to thank Dr Sophie Huczynska for answering many questions and latterly for spending considerable time reading and correcting. Also thanks to Murray Elder for his input.

I wish to thank my mother for her support and also for reading and checking this thesis. Also my grandparents for funding my accommodation throughout the year, without which this degree would have faltered in January. For funding my father is also to thank.

Abstract

A structure on sets of permutations based on involvement is defined, which forms a partial order on the set of all finite permutations. Sets with this structure we call closed classes, and we show how these can be expressed in terms of its Basis, or pattern avoidance set. Further structures on permutations and classes are introduced, in particular the notion of Atomic Classes.

We provide a list of constructions to form new classes from old, and, for each construction, discuss the existence of a Finite Basis for the new class, and the conditions required for this class to be Atomic. Particular emphasis is placed on the Wreath Product, where we exhibit the only known Finite Basis result.

A new method for proving the Finite Basis result for the wreath product $X \wr I$ is exhibited, and this is then adapted and applied to several other related wreath products to prove the Finite Basis Property in each case, all of which were previously unknown.

Chapter 1

Preliminaries

Permutations have been studied for a long time in a group theoretic context. More recently they have become the subject of a very different structure, which places emphasis on their shape and patterns within the permutation rather than their orbits and cycle types (although group properties have also been studied in this context [6]). Here we review and develop some of this structure theory. Research is also being carried out on enumeration, most recently with the proof of the Stanley-Wilf Conjecture [11].

We may motivate this new context with the following problem [9]. Suppose we have n children, no two of whom have the same height, and we wish to arrange them in a line so that each person faces the back of the preceding person. Additionally, each child must be able to see all the children that are shorter than him and are in front of him. What arrangements are allowed?

For $n = 8$ the lineup 15362478 would not work as, for example, 3 cannot see 1 since he cannot see over the top of 5. An example of a good lineup would be 85346721.

We may see that for a lineup to succeed, we cannot have three elements a, b, c in this order (although not necessarily consecutive), with $a < c < b$. If this were to occur, then b would not be able to see a.

1.1 Order Isomorphism and Involvement

Throughout this thesis we will refer to permutations using lowercase Greek letters (primarily σ, τ). Lowercase Roman letters (s, t etc.) will generally refer to a single symbol within a permutation, while uppercase letters (X, Y) will be used for sets and closed classes of permutations.

Definition 1.1.1. Two sequences $\sigma = [s_1, s_2, \ldots, s_n]$ and $\tau = [t_1, t_2, \ldots, t_n]$ of equal length n are said to be *order isomorphic*, $\sigma \cong \tau$, if for all i, j we have

$$s_i < s_j \Leftrightarrow t_i < t_j.$$

Given a sequence on n symbols, there is a unique ordering of the symbols $1, \ldots, n$ order isomorphic to it. Such an ordering on the symbols $1, \ldots, n$ will be called a *permutation*.

Example 1.1.2. The permutation $[5, 3, 6, 1, 2, 4]$ is order isomorphic to the sequence $[8, 4, 9, 1, 3, 7]$.

Definition 1.1.3. The permutation $\tau = [t_1, t_2, \ldots, t_m]$ is said to be *involved* in the permutation $\sigma = [s_1, s_2, \ldots, s_n]$, $\tau \preccurlyeq \sigma$, if τ is order isomorphic to a subsequence of σ. i.e. if there exists a subsequence $s_{i_1}, \ldots, s_{i_m}$ of σ such that $s_{i_1}, \ldots, s_{i_m} \cong \tau$.

If $\tau \ncong \sigma$ but $\tau \preccurlyeq \sigma$ then we may say that τ is *properly involved* in σ, and write $\tau \prec \sigma$.

For any permutation $\sigma = [s_1, \ldots, s_n]$, if $\tau = [t_1, \ldots, t_m]$ is a permutation with $m \leq n$ and τ is not involved in σ, then we say that σ *avoids* τ.

We note immediately that involvement is reflexive ($\sigma \preccurlyeq \sigma$ for all σ), transitive ($\sigma \preccurlyeq \tau$ and $\tau \preccurlyeq \rho$ implies $\sigma \preccurlyeq \rho$) and anti-symmetric ($\sigma \preccurlyeq \tau$ and $\tau \preccurlyeq \sigma$ implies $\sigma = \tau$) and hence forms a partial order on the set of all finite permutations.

Example 1.1.4. For example, $[3,1,2] \preccurlyeq [6,1,3,5,4,2]$ because of the subsequence $[6,3,4]$. Note that the involvement of one permutation within another is not necessarily unique.

On the other hand, $[6,1,3,5,4,2]$ avoids the permutation $[2,1,3]$.

When dealing with small permutations, we will often omit the brackets and commas. Thus the above example would be written $312 \preccurlyeq 653142$.

Since involvement is only a partial order, we can form lists of permutations that are not involved in each other. Such lists can be infinitely long, and they are of significance when considering the structure of closed classes, as we shall see shortly.

Definition 1.1.5. A set X of permutations is said to be an *antichain* if no two distinct elements of X are comparable under the partial order of involvement.

Example 1.1.6. The set $X = \{123, 312, 1432, 4321\}$ forms an antichain, as no pair, σ, τ of distinct elements from X has the property $\sigma \preccurlyeq \tau$ or $\tau \preccurlyeq \sigma$.

1.2 Closed Classes and Pattern Avoidance

Definition 1.2.1. A set X of permutations is said to be *closed* if, whenever $\sigma \in X$ and $\tau \preccurlyeq \sigma$, then $\tau \in X$.

If X is any set of permutations then the *closure* of X, denoted $\mathrm{Sub}(X)$, is the smallest closed set containing X. For example,

$$\mathrm{Sub}(\{1432, 1234\}) = \{1, 12, 21, 123, 132, 321, 1432, 1234\}.$$

Our use of the notation Sub() here is a particular case of a more general notation which will be described in Section 1.4.

An example of an infinite closed set is the set of *stack sortable* permutations [10]. Suppose we have a permutation $\sigma = [s_1, s_2, \ldots, s_n]$, then we want to know whether this permutation can be sorted into the trivial permutation $[1, 2, \ldots, n]$ by passing the permutation σ through a *stack*, symbol by symbol. The stack is an array holding any number of symbols vertically. Symbols are 'pushed' on to the top of the stack, and we may only ever remove, or 'pop', the topmost symbol. When we pop a symbol it is passed to the output. If this process of pushing and popping symbols from σ results in σ being sorted into the trivial permutation, then σ is stack sortable.

Example 1.2.2. 1. The permutation 54132 is stack sortable. We begin by pushing 5 and then 4 into the stack. The next symbol is 1, which we push and pop directly to the output. Next is 3, which we add onto the top of the stack above 4 and 5 (giving in the stack, top to bottom, 345). The remaining input symbol is 2 which we push and pop immediately, giving 12 in the output. Then we pop each of 3, 4 and 5 in turn to the output, giving the trivial permutation 12345, and hence 54132 is stack sortable.

2. The permutation 231 is not stack sortable. If we attempt to sort it, we must first push 2 into the stack. We then must push 3 on to the stack above 2. Finally symbol 1 can be pushed and popped immediately. But now we must pop symbol 3 and then 2, which gives us the output permutation 132.

In fact 231 is the smallest non stack sortable permutation and we have the following lemma [10].

Lemma 1.2.3. *A permutation σ is stack sortable if and only if it does not involve the permutation* 231*.*

Proof. Let $\sigma = s_1, \ldots, s_n$ be a permutation, and suppose we have a 231-pattern in σ formed by the entries s_i, s_j and s_k (so that $i < j < k$ and $s_k < s_i < s_j$).

Suppose we have been able to sort σ. As $i < j$ and $s_i < s_j$ we must have popped s_i off the stack before s_j was pushed on. As $j < k$ but $s_j > s_k$, we must have left s_j on the stack until after s_k was pushed on. Similarly s_i must also have been left on until after s_k was pushed on. This is impossible, as s_i was popped off before s_j was pushed on, but we need both to remain on the stack until after s_k has appeared, a contradiction.

Now suppose that σ does not involve 231. We define the algorithm for sorting σ inductively as follows. We say that at every stage i we have output the symbols $1, \ldots, i-1$ from the stack in increasing order. Two cases for the position of the next symbol i arise:

1. i is still in the input queue. We proceed by adding symbols into the stack until we encounter i, and pass this straight to the output.

2. i is already in the stack. Then note that in σ the symbol $i-1$ must have been placed to the right of all the symbols currently in the stack, and in particular symbol i in σ. Suppose there is a $j > i$ on the top of the stack, then in our original permutation σ we would have i preceding j, and both of these precede symbol $i-1$, giving a 231-pattern. So no such j can exist, and thus symbol i is on top and we can move it straight to the output. □

Bóna [9] defined a stack differently – one where the symbols in the stack must be in the order smallest on top to largest at the bottom (often called an ordered stack) – but yet we still obtain the same result: the permutations which are sortable in this arrangement are precisely those which avoid the pattern 231. In the unordered case if we have symbols $j > i$ with j above i in the stack, then i could never be output without first removing j, in which case the output would not be in increasing order. Thus stack sortable permutations are always passed through ordered and unordered stacks in the same way.

Differences arise if we start arranging stacks in series. For example, the permutation 2341 can be sorted using two unordered stacks by pushing all symbols into the first stack, then popping symbol 1 directly to the output, and symbols 4,3 and 2 on to the second stack, from where they can be removed in increasing order. However, 2341 cannot be sorted on ordered stacks, as we will end up with 2 on the second stack, 3 on the first and 4 waiting to be pushed, which can only be done after popping 2 to the output and 3 on to the second stack.

Both these stacks are examples of *permuting machines* – a machine M that accepts any finite input sequence of objects (normally $1, 2, \ldots$) and produces a permutation of these objects [5].

The class X of permutations that are stack sortable is clearly closed. Noticing that X is composed of exactly those permutations which *avoid* the pattern 231 motivates the following important concept.

Definition 1.2.4. Let X be a closed class. The set of permutations, minimal with respect to involvement, that do not lie in X is called the *basis* of X, and is written $\mathcal{B}(X)$.

Conversely, let B be a basis of permutations. The *avoidance class* of B is the unique closed set of permutations which do not involve any elements of B. This is written $\mathcal{A}(B)$.

Thus the class associated with a stack is written $\mathcal{A}(231)$, but examples are not restricted to those described by permuting machines. The following are taken from [4, 8] and we will meet many of them again in later parts of this thesis.

Example 1.2.5.

1. The set $I = \{1, 12, 123, \ldots\}$ of identity permutations has basis $\{21\}$. This set is also the closed class associated with the 'queue' permuting machine – a machine that can store symbols, but cannot reorder them. Similarly the set $R = \{1, 21, 321, \ldots\}$ of reversed

permutations has basis $\{12\}$.

2. The set X which can be expressed as the interleaving of two increasing subsequences has basis $\{321\}$ [4].

3. The set S of 'separable' permutations – those which can be expressed as a direct or skew sum (see Section 1.3) of two smaller permutations, both of which must also be separable, or trivial – has basis $\{3142, 2413\}$.

4. The set X of permutations obtained by a 'riffle' shuffle of a deck of cards $1, 2, \dots, n$ (all permutations consisting of two increasing sequences interleaved) has basis $\{321, 2143, 2413\}$ by Proposition 3.4 in [4].

It is easy to see that every basis B must be an antichain, and conversely every antichain is a basis for some closed class. All the above examples have finite bases, but as we can form infinite antichains, we can have closed classes whose bases are infinite. From [12]:

Example 1.2.6. The infinite set $A = \{\alpha_i : i \in \mathbb{N}\}$ of permutations defined by:

$$\begin{aligned}
\alpha_1 &= [3, 2, 5, 1, 9, 4, 8, 6, 7] \\
\alpha_2 &= [3, 2, 5, 1, 7, 4, 11, 6, 10, 8, 9] \\
\alpha_3 &= [3, 2, 5, 1, 7, 4, 9, 6, 13, 8, 12, 10, 11] \\
\alpha_4 &= [3, 2, 5, 1, 7, 4, 9, 6, 11, 8, 15, 10, 14, 12, 13] \\
&\vdots \\
\alpha_n &= [3, 2, 5, 1, 7, 4, 9, 6, \dots \\
&\qquad \dots, 2n+3, 2n, 2n+7, 2n+2, 2n+6, 2n+4, 2n+5] \\
&\vdots
\end{aligned}$$

is an antichain, and hence the class $\mathcal{A}(A)$ is a closed class with infinite basis.

Showing that A is an antichain is itself not immediately clear and we refer the reader to Chapter 3 of [12].

We now notice the following trivial properties of avoidance and the basis:

- We may extend the definition of pattern avoidance to any set of permutations A, forming the closed class $\mathcal{A}(A)$. If A is not an antichain then there exists an antichain $B \subseteq A$ such that $\mathcal{A}(A) = \mathcal{A}(B)$. In this case we have $\mathcal{B}(\mathcal{A}(A)) = B$.

  If A is an antichain then it is also the basis for $\mathcal{A}(A)$, and hence $\mathcal{B}(\mathcal{A}(A)) = A$.

- For any closed class X, we observe that the avoidance class of its basis is itself: $\mathcal{A}(\mathcal{B}(X)) = X$.

For this thesis, we will devote a lot of attention to the problem of classes with finite basis, and whether this finite basis property holds as we construct new classes from old.

1.3 Permutation and Class Types

Here we review several properties that permutations and classes can possess. We begin with four pieces of terminology that we will encounter frequently later on.

Definition 1.3.1. When describing subsequences of a permutation $\sigma = [s_1, \ldots, s_n]$, by *segment* we mean a subsequence of σ whose entries occur consecutively, i.e. any subsequence order isomorphic to $[s_i, s_{i+1}, \ldots, s_{j-1}, s_j]$ for some i, j with $i \leq j$.

The term *proper segment* of the permutation $\sigma = [s_1, \ldots, s_n]$ will be used for segments of length k with $1 < k < n$.

Definition 1.3.2. Two symbols s, t of a permutation σ are *adjacent* if $[s, t]$ is a segment of σ.

Definition 1.3.3. Two symbols s, t of a permutation σ are *consecutive increasing* (resp. *decreasing*) if s and t are adjacent with $s < t$ (resp. $s > t$) and there is no symbol q such that $s < q < t$ (resp. $s > q > t$). We may extend this in the obvious way to segments of length n.

Definition 1.3.4. Let $\sigma = [s_1, \ldots, s_n]$ be any permutation. An *interval* of σ is a segment $[s_i, s_{i+1}, \ldots, s_j]$ for some i, j with $i \leq j$ whose symbols can be rearranged to form a consecutive increasing segment.

If the interval $[s_i, s_{i+1}, \ldots, s_j]$ is a proper segment of σ then it is a *proper interval*.

Example 1.3.5. Let $\sigma = 132456$.

- The symbols 1 and 3 are adjacent but not consecutive increasing because $1 < 2 < 3$.
- The segment 32 is consecutive decreasing.
- The segments 45 and 56 are both consecutive increasing pairs. We also have that 456 is a consecutive increasing segment.
- σ has numerous intervals. Some proper intervals are 132, 1324, 3245 and 456.

Definition 1.3.6. We define *deletion* of a symbol s_i from a permutation $\sigma = [s_1, s_2, \ldots, s_n]$ to be the permutation $\sigma - s_i$ of length $n - 1$ order isomorphic to the sequence $[s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n]$.

Example 1.3.7. If $\sigma = [7, 3, 6, 2, 5, 1, 4]$ then deleting the symbol '4' gives $\sigma - 4 = [6, 3, 5, 2, 4, 1]$.

We now exhibit three constructions of permutations from other permutations. These will all be extended to constructions of classes of permutations in Chapter 2.

Definition 1.3.8. The permutation $\pi = [p_1, p_2, \ldots, p_{n+m}]$ is a *concatenation* of the permutations $\sigma = [s_1, \ldots, s_n]$ with $\tau = [t_1, \ldots, t_m]$ if the segment $p_1, \ldots, p_n$ is order isomorphic to σ and $p_{n+1}, \ldots, p_{n+m}$ is order isomorphic to τ.

The set of concatenations of σ with τ will be denoted $\sigma\tau$.

Two special cases of concatenation are the direct and skew sums of two permutations. First we make a note about notation:

If σ and τ are two sequences, then we will write $\sigma < \tau$ if $s < t$ for all $s \in \sigma$ and $t \in \tau$. We define $\sigma > \tau$ analogously.

Definition 1.3.9. The *direct sum* $\sigma \oplus \tau$ of two sequences $\sigma = [s_1, \ldots, s_n]$ and $\tau = [t_1, \ldots, t_m]$ is the unique concatenation $\pi = [p_1, \ldots, p_{n+m}] \in \sigma\tau$ with $[p_1, \ldots, p_n] < [p_{n+1}, \ldots, p_{n+m}]$.

Similarly, the *skew sum* $\sigma \ominus \tau$ of two permutations σ and τ is the unique concatenation $\pi \in \sigma\tau$ with $[p_1, \ldots, p_n] > [p_{n+1}, \ldots, p_{n+m}]$.

Example 1.3.10. Let $\sigma = 3142$ and $\tau = 231$ then their direct sum is $\sigma \oplus \tau = 3142675$. Similarly, their skew sum is $\sigma \ominus \tau = 6475231$.

We may now define a notion of permutations that cannot be expressed as a direct or skew sum [8].

Definition 1.3.11. A permutation σ is *forward* (resp. *backward*) *indecomposable* if it cannot be expressed as a direct (resp. skew) sum $\sigma = \alpha \oplus \beta$ (resp. $\alpha \ominus \beta$). If σ is both forward and backward indecomposable it is said to be *strongly indecomposable.*

If a permutation σ is not forward (resp. backward) indecomposable then we can write it as a *sum decomposition* (resp. *skew decomposition*) , $\sigma = \alpha \oplus \beta$ (resp. $\sigma = \alpha \ominus \beta$) for some α, β.

Example 1.3.12.

1. The permutation $14253 = 1 \oplus 3142$ and so is forward decomposable. However it cannot be written as a skew sum and

hence is backward indecomposable.

2. The permutation $43521 = 213 \ominus 21$ and so is not backward indecomposable. It is, however, forward indecomposable.

3. The permutation 43152 cannot be expressed either as a direct sum or a skew sum, and hence is strongly indecomposable.

Notice that a permutation must be at least one of forward or backward indecomposable: it cannot be expressed both as a direct sum and a skew sum. We will extend this idea of indecomposability when discussing simple permutations in Subsection 1.3.1 – there we will look at permutations which cannot be written as mixtures of direct and skew sums.

Definition 1.3.13. A permutation σ is *irreducible* if it contains no segment of the form $i, i+1$. Similarly a permutation that has no segment of the form $i+1, i$ is *skew irreducible*. If it has neither a segment $i, i+1$ nor a segment $i+1, i$ then it is called *strongly irreducible*.

Example 1.3.14. 1. The permutation 31254 is neither irreducible nor skew irreducible, because of the segments 12 and 54 respectively.

2. The permutation 12534 is not irreducible because of the segments 12 and 34, but it is skew irreducible.

3. The permutation 15324 is irreducible as it has no increasing segments, but it is not skew irreducible because of the segment 32.

4. The permutation 13524 is strongly irreducible as we have no increasing or decreasing segments.

Definition 1.3.15. A set X of permutations is *sum complete* or just *complete* if for all $\sigma, \tau \in X$ we also have $\sigma \oplus \tau \in X$. It is *skew complete* if for all $\sigma, \tau \in X$ we have $\sigma \ominus \tau \in X$.

It is *strongly complete* if X is both sum and skew complete.

Definition 1.3.16. A set X of permutations is *expanded* if for any permutation $\sigma = \alpha i \beta \in X$, the permutation obtained by replacing i by the segment $i, i+1$ and replacing all symbols $s \in \sigma$, $s > i$, by $s+1$, also lies in X. This permutation is called the *expansion* of σ at i.

Similarly, replacing i by $i+1, i$ is the *negative* (or *skew*) *expansion*, and so a set X is *skew expanded* if the skew expansion of any permutation $\sigma \in X$ at any symbol i also lies in X.

The set X is *strongly expanded* if it is invariant under both the expansion and the skew expansion.

Example 1.3.17. The set I of identity permutations is expanded. Similarly the set R of reverse permutations is skew expanded.

For any closed class X, we can now define the *completion* (resp. *strong completion*, *expansion*, *strong expansion*) to be the smallest closed class containing X that is complete (resp strongly complete, expanded, strongly expanded).

Example 1.3.18.

1. The set of identity permutations I (resp. separable permutations S) is both the completion and expansion (resp. strong completion and strong expansion) of the set consisting of the trivial permutation $T = \{1\}$ [8].

2. The set $X = \mathcal{A}(231)$ of stack-sortable permutations is sum complete. If $\sigma, \tau \in X$ then the element $\sigma \oplus \tau$ can clearly be stack sorted by first sorting σ and then τ. It is also skew expanded, for if we skew expand any permutation $\sigma \in X$ at symbol i, we cannot introduce any instance of the pattern 231.

The following lemma is from [8] and gives a connection between particular closed classes and the form of their basis elements.

Lemma 1.3.19. *1. The closed set X is expanded (respectively, strongly expanded) if and only if every basis element is irreducible (resp. strongly irreducible).*

2. The closed set X is complete (respectively, strongly complete) if and only if every basis element is indecomposable (resp. strongly indecomposable).

Proof. 1. Suppose X is expanded. If any $\beta \in \mathcal{B}(X)$ has a segment $i, i+1$, then by minimality $\beta - (i+1) \in X$ and so β is the expansion of an element in X. Since X is expanded this means $\beta \in X$, a contradiction.

Conversely, suppose every $\beta \in \mathcal{B}(X)$ is irreducible. For any $\sigma \in X$, let σ_i be the permutations formed by expanding σ at i. Suppose, for some $\beta \in \mathcal{B}(X)$ and some symbol i, that $\beta \preccurlyeq \sigma_i$. Then as β is irreducible we must have $\beta \preccurlyeq \sigma$, a contradiction. Hence $\sigma_i \in X$ for all i and so X is expanded.

2. Suppose X is complete. If any $\beta \in \mathcal{B}(X)$ can be written as a direct sum $\beta = \sigma \oplus \tau$, then by minimality $\sigma, \tau \in X$ and so β is the direct sum of two permutations in X. Since X is complete this means $\beta \in X$, a contradiction.

Conversely, suppose every $\beta \in \mathcal{B}(X)$ is irreducible. For any $\sigma \in X$, let σ_i be the permutations formed by expanding σ at i. Suppose, for some $\beta \in \mathcal{B}(X)$ and some symbol i, that $\beta \preccurlyeq \sigma_i$. Then as β is irreducible we must have $\beta \preccurlyeq \sigma$, a contradiction. Hence $\sigma_i \in X$ for all i and so X is expanded.

Conversely, suppose every $\beta \in \mathcal{B}(X)$ is indecomposable. Suppose, for some $\beta \in \mathcal{B}(X)$ and $\sigma, \tau \in X$, that $\beta \preccurlyeq \sigma \oplus \tau$. Then as β is indecomposable we must have $\beta \preccurlyeq \sigma$ or $\beta \preccurlyeq \tau$, a contradiction.

The versions with X strongly expanded or strongly complete are similar.

□

In fact, we can also easily extend this lemma to skew expanded and skew complete, where all basis elements of such a class would have to be skew irreducible and backward indecomposable respectively.

1.3.1 Simple Permutations

Permutations of the following type are discussed in [1, 2], where their presence within a closed class can be used to give more information about the enumeration of elements in the class, and indeed about elements in the basis.

Definition 1.3.20. A *simple permutation* is a permutation in which no proper segment is mapped to a proper interval. For example, 2647513 is not simple as it maps 2345 onto 4567.

From [1] we have the following result which is of interest to our review of finitely based classes. We present it here without proof.

Proposition 1.3.21. *Every closed class of permutations having only a finite number of simple permutations has a finite basis.*

The permutations 1, 12 and 21 are simple but are something of a special case, and there are no simple permutations of length 3. For simple permutations of length ≥ 4, it is clear from the definition that they are both strongly indecomposable and strongly irreducible. The converse is not true - for example the permutation 5724136 is not simple because the interval 3456 maps onto the interval 1234, but yet we cannot form a sum or skew decomposition.

By Lemma 1.3.19 we have that any closed class whose basis elements are all simple must be both strongly expanded and strongly complete. We will encounter simple permutations again when discussing properties of the wreath product, in particular with respect to wreath closure.

1.4 Sub() Representation and Atomic Classes

We recall the Sub() notation that was introduced in Definition 1.2.1 to mean the closure of a set of permutations. This is in fact a special case of a more general notation introduced in [7, 12]. Suppose we have two sets A and B of real numbers, and let π be an injection from A to B. Then every finite subset $a_1, a_2, \ldots, a_n$ of A, with $a_1 < a_2 < \ldots < a_n$, maps to a sequence $\pi(a_1), \pi(a_2), \ldots, \pi(a_n)$ of B. Such a sequence is order isomorphic to some permutation. We denote the set of permutations that arise in this way as $\mathrm{Sub}(\pi : A \to B)$. It is easy to see that such a set of permutations is closed.

Often the domain and range of π are clear from the context, and so we simply write $\mathrm{Sub}(\pi)$. Moreover, we can always replace B by the range of π, and so we always assume that π is a bijection $A \to B$.

So for a single finite permutation σ of length n, $\mathrm{Sub}(\sigma)$ as introduced in Definition 1.2.1 is shorthand for $\mathrm{Sub}(\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\})$. We can, of course, have an injection or permutation with infinite domain and/or range, in which case our notion of closure is the same, except that the resulting closed class will have infinitely many permutations. The following example is taken from [7].

Example 1.4.1. Let $A = \{1-1/2^i, 2-1/2^i | i = 1, 2, \ldots\}$ and $B = \{1, 2, \ldots\}$. Define π by:

$$\pi(x) = \begin{cases} 2i - 1 & \text{if } x = 1 - 1/2^i \\ 2i & \text{if } x = 2 - 1/2^i. \end{cases}$$

We can check that any finite increasing sequence of elements in A maps to an increasing sequence of odd integers followed by an increasing sequence of even integers. Thus the permutations in $\mathrm{Sub}(\pi : A \to B)$ are all made from concatenations of two increasing segments. [4] tells us this has basis $\{321, 2143, 3142\}$.

A closed set X which can be expressed in the form $\mathrm{Sub}(\pi : A \to B)$ is one possible definition of an atomic class. There are, however, several equivalent conditions [7].

Theorem 1.4.2. *The following conditions on a closed set X are equivalent:*

1. $X = \mathrm{Sub}(\pi : A \to B)$ *for some sets A, B and bijection π.*

2. *X cannot be expressed as a union of two proper closed subsets.*

3. *For every $\sigma, \tau \in X$ there exists γ such that $\sigma \preccurlyeq \gamma$ and $\tau \preccurlyeq \gamma$ (the* Join Property*).*

4. *X contains permutations $\gamma_1 \preccurlyeq \gamma_2 \preccurlyeq \ldots$ such that for every $\sigma \in X$ we have $\sigma \preccurlyeq \gamma_n$ for some n.*

Definition 1.4.3. A class X obeying any (and hence all) of 1-4 in the above theorem is said to be *atomic*.

Expressing a class as a union of two classes is often a useful way to demonstrate non-atomicity. This has been studied in depth in [12].

Example 1.4.4. In [4] it is shown that $\mathcal{A}(321, 2143) = \mathcal{A}(321, 2143, 3142) \cup \mathcal{A}(321, 2143, 2413)$.

Note that any class that avoids a single permutation must be atomic. We formalise this as a proposition.

Proposition 1.4.5. *$\mathcal{A}(\beta)$ is atomic for any permutation β.*

Proof. We use the Join Property: take any $\sigma, \tau \in \mathcal{A}(\beta)$. By comments in Section 1.3 β must be forward or backward indecomposable (or both), since it cannot be expressed both as a direct sum and a skew sum.

- If β is forward indecomposable then the element $\sigma \oplus \tau$ lies in $\mathcal{A}(\beta)$. To see this, suppose $\beta \preccurlyeq \sigma \oplus \tau$. As β is not involved in either σ or τ, we must have the concatenation $\beta = \beta_1\beta_2$ with $\beta_1 \preccurlyeq \sigma$ and $\beta_2 \preccurlyeq \tau$. Moreover, by the construction of $\sigma \oplus \tau$, we must have the direct sum $\beta = \beta_1 \oplus \beta_2$, a contradiction. Thus $\sigma, \tau \preccurlyeq \sigma \oplus \tau \in \mathcal{A}(\beta)$.

- Similarly, if β is backward indecomposable then the skew sum $\sigma \ominus \tau$ lies in $\mathcal{A}(\beta)$. $\square$

In the next chapter we will investigate the atomicity of various new classes that are constructed from old. In particular we are interested in whether our constructions from atomic classes result in new atomic classes.

1.4.1 Natural Classes

Definition 1.4.6. A *natural class* is a closed set of the form

$$\mathrm{Sub}(\pi : \mathbb{N} \to \mathbb{N}).$$

That is, an atomic class based on a permutation π of the natural numbers.

These have been studied in [7, 12], specifically in their relation to complete classes. They are also closely linked to derivatives, which will be discussed in Section 2.6. Atkinson *et al* [7] present a complete characterisation of finitely based natural classes as follows.

Proposition 1.4.7. *Let X be a finitely based natural class. Then either*

1. *$X = \mathrm{Sub}(\gamma) \oplus S$ (direct sum of two sets: see Section 2.3) where γ is a finite permutation and S is a sum complete closed class determined uniquely by X, or*

2. *$X = \mathrm{Sub}(\pi)$ where π is unique and ultimately periodic in the sense that there exist integers N and $P > 0$ such that, for all $n \geq N$, $\pi(n + P) = \pi(n) + P$.*

Chapter 2

Constructions

In this chapter we review a number of methods of constructing closed classes, and pay particular attention to the format of the basis elements, which leads to results on finitely-based classes.

We will cover both the *finite basis property* (constructions where our original class being finitely based implies the new class is also finitely based) and atomicity questions for each construction. In some cases these properties are straightforward but in others they are less clear, or even unknown.

2.1 Intersection

We can consider the intersection of two sets of permutations in the usual set theoretical way. Closure and the finite basis property are straightforward [4].

Theorem 2.1.1. *Let X and Y be closed classes of permutations. Then $X \cap Y$ is also closed. Moreover, if both X and Y have finite basis, then so does $X \cap Y$.*

Proof. That $X \cap Y$ is closed is trivial. Now suppose that $A = \mathcal{B}(X)$ and $B = \mathcal{B}(Y)$ are finite. That $X \cap Y$ has a finite basis comes directly from the observation that $X \cap Y = \mathcal{A}(A \cup B)$:

- If $\sigma \in X \cap Y$ then since $\sigma \in X$ it must avoid all permutations in A. Similarly $\sigma \in Y$ tells us it avoids all permutations in B, and so $\sigma \in \mathcal{A}(A \cup B)$. Hence $X \cap Y \subseteq \mathcal{A}(A \cup B)$.

- Conversely, if $\sigma \in \mathcal{A}(A \cup B)$ then σ avoids all permutations in both A and B, which tells us $\sigma \in \mathcal{A}(A)$ and $\sigma \in \mathcal{A}(B)$. Thus $\sigma \in X$ and $\sigma \in Y$, so $\sigma \in X \cap Y$. □

Atomicity In general, the intersection of two atomic classes need not be atomic. For example, consider $\mathcal{A}(321, 2143)$, which may be written as the intersection of the two atomic classes $\mathcal{A}(321) \cap \mathcal{A}(2143)$. However, we have already seen that it can also be written as the union of two proper closed classes, $\mathcal{A}(321, 2143, 3142) \cup \mathcal{A}(321, 2143, 2413)$, and hence is not atomic by Proposition 1.4.5.

2.2 Union

Theorem 2.2.1. *Let X and Y be closed classes of permutations. Then $X \cup Y$ is also closed, and if both X and Y have finite basis then so does $X \cup Y$.*

Proof. $X \cup Y$ is clearly closed. Let $A = \mathcal{B}(X)$ and $B = \mathcal{B}(Y)$ be finite. Consider a permutation $\gamma \in \mathcal{B}(X \cup Y)$. Such a permutation cannot lie in X or Y and so has subsequences α and β which are order isomorphic to elements in A and B respectively.

As such a permutation is minimal, no proper subsequence of γ can have this property, so γ must be the union of α and β and hence there are only finitely many possibilities for γ. □

In fact, we can say exactly what the basis elements are in terms of A and B. In Section 2.5 we will introduce the notion of *merges*, and in [12] we are introduced to the idea of a *minimal merge*: minimal in the sense that our merge of two sets $A \sqcup B$ is minimal under involvement. The basis for $X \cup Y$ is then just the minimal merge of $\mathcal{B}(X)$ and $\mathcal{B}(Y)$.

Example 2.2.2. Let $X = I = \mathcal{A}(21)$, $Y = R = \mathcal{A}(12)$. Then the basis of $I \cup R$ is the minimal merge of 21 and 12, which we can easily see to be $\{213, 312, 231, 132\}$.

It is clear that the union of two classes cannot be atomic unless one is contained within the other, as otherwise we would immediately violate condition 2 of Theorem 1.4.2.

2.3 Direct and Skew Sums

Recall our definition of direct and skew sums of permutations 1.3.9. We may extend this to sets in an obvious way,

$$X \oplus Y = \{\sigma \oplus \tau : \sigma \in X, \tau \in Y\}$$

and it is clear that if X and Y are closed classes then so is $X \oplus Y$.

Similarly skew sum extends in the obvious manner to sets $X \ominus Y$, and again we see that if X and Y are closed, then so is $X \ominus Y$.

However, the direct or skew sum of two finitely based closed classes X and Y need not be finitely based. We have the following example that is mentioned in Section 2.2.4 of [12]. There the class X is defined in error to be $\mathcal{A}(321)$, whose sum completion is itself,

$$\mathcal{A}(321) \oplus \mathcal{A}(321) = \mathcal{A}(321)$$

and hence the direct sum is finitely based. Here we correct this error, noting that what is meant is $X = \mathcal{A}(321654)$ and exhibit infinitely many basis elements of $X \oplus X$.

Example 2.3.1. We will see in Section 2.8.2 that the completion of the class $X = \mathcal{A}(321654)$ is not finitely based. [8] does this by exhibiting an infinite antichain that lies in the basis of the completion of X. Here, we use the same antichain, and show that it also lies in the basis of $X \oplus X$.

Consider the permutations β_m, for $m > 2$, defined by:

$$\begin{aligned} \beta_m &= 3,2,5,1,7,4,9,6,11,8,\ldots,2i-1,2i-4,2i+1,2i-2,\ldots \\ &\quad 2m-1,2m-4,2m+2,2m-2,2m+1,2m \end{aligned}$$

Apart from the first four and last four symbols, β_m consists of interleaving odd-valued and even-valued symbols, as indicated by the segment $2i-1, 2i-4, 2i+1, 2i-2$. Next observe that the subsequence $3, 2, 1, 2m+2, 2m+1, 2m$ of β_m is order isomorphic to 321654, and in fact is the only subsequence of β_m with this property. In particular, $\beta_m \notin X$. Observe further that the segment 3251 overlaps 5174, which in turn overlaps 7496, etc., and so β_m is indecomposable. In particular, we cannot express it as a direct sum, and so $\beta_m \notin X \oplus X$.

We now consider the effect of omitting any one symbol from β_m. If we omit one of $3, 2, 1, 2m+2, 2m+1$ or $2m$ then we no longer have the subsequence 321654, and so what results lies in X. As $X \subseteq X \oplus X$ it also lies in $X \oplus X$.

If we omit any other symbol x, say, observe that the resulting permutation is decomposable, and so we may write $\beta_m - x = \sigma \oplus \tau$. Now we note that 321 is involved in both σ and τ, but 321654 occurs in neither, and hence $\sigma, \tau \in X$. Thus $\beta_m - x \in X \oplus X$.

Thus $\beta_m \notin X$ but $\beta_m - x \in X \oplus X$ for all $1 \leq x \leq |\beta_m|$, and so β_m is minimally not in $X \oplus X$. Hence $\beta_m \in \mathcal{B}(X \oplus X)$, for all $m > 2$.

Atomicity Murphy states in [12] that the direct sum and skew sum of two atomic classes is also atomic. We supply the proof below.

Proposition 2.3.2. *The direct and skew sum of two atomic classes is also atomic.*

Proof. Let X and Y be two atomic classes. We use the Join Property of atomic classes (condition 3 of Theorem 1.4.2).

Consider any two permutations $\sigma, \tau \in X \oplus Y$. Then each can be expressed as a direct sum, $\sigma = \alpha_1 \oplus \beta_1$, $\tau = \alpha_2 \oplus \beta_2$ with $\alpha_1, \alpha_2 \in X$ and $\beta_1, \beta_2 \in Y$.

Since X and Y are atomic, there exists an $\alpha \in X$ and a $\beta \in Y$ such that $\alpha_1, \alpha_2 \preccurlyeq \alpha$ and $\beta_1, \beta_2 \preccurlyeq \beta$. Then the direct sum $\alpha \oplus \beta$ lies in $X \oplus Y$ and contains both σ and τ. Hence $X \oplus Y$ is atomic.

The proof for skew sum is similar. □

2.4 Juxtaposition

Definition 2.4.1. Let X and Y be sets of permutations. Then their *juxtaposition* $[X, Y]$ is the set of all permutations which can be expressed as concatenations $\sigma\tau$ with $\sigma \in X$ and $\tau \in Y$.

The juxtaposition of two closed classes has been studied in [3, 4, 12]. Notice that both direct and skew sums are just special cases of juxtaposition - in fact, for two permutations σ and τ of lengths m and n respectively there are $\binom{m+n}{n}$ distinct juxtapositions [12].

The following theorem by Atkinson [4] gives us information about the finite basis property for juxtaposition, and hence also for the direct and skew sums. The proof also indicates the form that elements in the basis must take.

Theorem 2.4.2. *Let X and Y be closed sets. Then their juxtaposition $[X, Y]$ is also closed. Moreover if X and Y are finitely based then so is $[X, Y]$.*

Proof. That $[X, Y]$ is closed is straightforward. Suppose X and Y are finitely based, and take $\alpha \in \mathcal{B}([X, Y])$. We write $\alpha = \sigma\tau k$ where k is the last symbol of α and we may assume σ and τ are order isomorphic to permutations in

X and Y respectively (since α is minimal with respect to not belonging to $[X, Y]$). We choose this decomposition such that σ is of maximal length.

If t is the first symbol of τ then, as σ was maximal, we must have σt not being order isomorphic to a permutation of X. We also require τk not to be order isomorphic to any permutation in Y as otherwise $\sigma(\tau k)$ is a concatenation of an element in X and an element in Y, and hence in $[X, Y]$.

Thus σt must have a subsequence $\sigma' t$ which is order isomorphic to a permutation in $\mathcal{B}(X)$, and similarly τk has a subsequence $\tau' k$ order isomorphic to a permutation in $\mathcal{B}(Y)$. But then the element from the concatenation, $\sigma' t \tau' k$, of these two basis elements which is involved in α (or just $\sigma' \tau' k$ if t is the first element in τ'), cannot be in $[X, Y]$ and so by minimality $\sigma' t \tau' k = \alpha$.

Since X and Y are finitely based we have an upper bound on the lengths of $\sigma' t$ and $\tau' k$, and thus a bound on the length of α. □

This proof tells us that basis elements of $[X, Y]$, with X and Y both finitely based, are either a juxtaposition $\alpha\beta$ with $\alpha \in \mathcal{B}(X)$, $\beta \in \mathcal{B}(Y)$, or there is an overlap of *at most* one element - being the last element of α and the first of β.

Example 2.4.3. The juxtapositions $[I, R, I, R, \ldots]$ are studied in depth in [3]. Taking just two of these $X = [I, R]$, recall that $I = \mathcal{A}(21)$ and $R = \mathcal{A}(12)$. By the theorem, we only need to look for basis elements of length 3 or 4, and in particular only those first consisting of a decreasing segment, then an increasing one (not necessarily consecutive). Thus the only candidates are:

$$213, 312, 2134, 3124, 3214, 4123, 4213, 4312$$

but notice that all the permutations in this list of length 4 involve at least one of 213 and 312, and so we have $X = \mathcal{A}(213, 312)$.

Atomicity The juxtaposition of two atomic classes need not be atomic. As a counter-example, we have [12]:

Example 2.4.4. Let $X = Y = \{\varnothing, 1\}$, the atomic class consisting of the empty permutation and the trivial permutation. Then their juxtaposition $[X, Y] = \{\varnothing, 1, 12, 21\}$ has both the permutations 12 and 21, but no permutation in $[X, Y]$ involves both of these, so $[X, Y]$ does not possess the Join Property.

We also have an example where the juxtaposition of two classes is atomic [12]:

Lemma 2.4.5. *If* $Y = \text{Sub}(213456\ldots)$*, then the juxtaposition* $[I, Y]$ *is atomic.*

Proof. We use the Join Property (condition 3 of Theorem 1.4.2). Let $Z = [I, Y]$ with Y as in the statement of the proposition. Note that any permutation in Y is either an identity permutation $[1, 2, \ldots, n]$ or a permutation of the form $[2, 1, 3, 4, \ldots, n]$ for some n. This latter type we will refer to as a *tick* permutation.

Any element of Z can be considered as being composed of two segments: an increasing segment order isomorphic to a permutation in I and a segment order isomorphic to a permutation in Y. We will write the symbols of any $\sigma \in Z$ in the following way:

$$\sigma = (s_{11} \ldots s_{1i_1})(s_{21} \ldots s_{2i_2}) \ldots (s_{n+1,1} \ldots s_{n+1,i_{n+1}})s_1 s_2 \ldots s_n$$

where the segment $s_{11} \ldots s_{ni_n}$ is from I and $s_1 \ldots s_n$ is order isomorphic to a permutation from Y. Moreover for each $3 < j < n+1$, we arrange that the small bracketed segments $s_{j1} \ldots s_{ji_j}$ are consecutive increasing, and lie between s_{j-1} and s_j. The segment $s_{n+1,1} \ldots s_{n+1,i_{n+1}}$ corresponds to the consecutive increasing sequence lying above all the final entry s_n.

The segments $s_{j1} \dots s_{ji_j}$ with $1 \leq j \leq 3$ are also consecutive increasing, and their position is determined by whether $s_1 \dots s_n$ is an identity or a tick permutation:

- Identity. Then $s_1 s_2 s_3 \cong 123$ and
$$(s_{11} \dots s_{1i_1}) < s_1 < (s_{21} \dots s_{2i_2}) < s_2 < (s_{31} \dots s_{3i_3}) < s_3.$$

- Tick. Then $s_1 s_2 s_3 \cong 213$ and
$$(s_{11} \dots s_{1i_1}) < s_2 < (s_{21} \dots s_{2i_2}) < s_1 < (s_{31} \dots s_{3i_3}) < s_3.$$

Now we consider any $\sigma, \tau \in Z$, and write

$$\begin{aligned} \sigma &= (s_{11} \dots s_{1i_1})(s_{21} \dots s_{2i_2}) \dots (s_{n+1,1} \dots s_{n+1,i_{n+1}}) s_1 s_2 \dots s_n \\ \tau &= (t_{11} \dots t_{1j_1})(t_{21} \dots t_{2j_2}) \dots (t_{m+1,1} \dots t_{m+1,j_{m+1}}) t_1 t_2 \dots t_m. \end{aligned}$$

We construct a new element π of Z, whose structure depends on the permutation types of $s_1 \dots s_n$ and $t_1 \dots t_m$. We write π in the same form as σ and τ:

$$\pi = (p_{11} \dots p_{1k_1})(p_{21} \dots p_{2k_2}) \dots (p_{q+1,1} \dots p_{q+1,k_{q+1}}) p_1 p_2 \dots p_q$$

where the parameters q and $k_1, \dots, k_{q+1}$, and the structure of $p_1 \dots p_q$ are determined according to one of the following 3 cases:

1. Both $s_1 \dots s_n$ and $t_1 \dots t_m$ are identity permutations. Then $p_1 \dots p_q$ is an identity permutation with $q = \max(m, n)$. We take $k_l = \max(i_l, j_l)$ for all $1 \leq l \leq \min(m, n) + 1$, and thereafter $k_l = i_l$ or $k_l = j_l$ for $\min(m, n) + 1 < l \leq q + 1$ dependent on which of m or n was larger.

2. $s_1 \dots s_n$ is a tick and $t_1 \dots t_m$ is an identity permutation. Then $q = \max(n, m-2)$, $p_1, \dots, p_q$ is a tick permutation and:
$$k_l = \begin{cases} i_l & \text{for } l = 1, 2 \\ \max(i_l, j_{l-2}) & \text{for } 3 \leq l \leq \min(n, m-2) + 1 \\ i_l & \text{for } \min(n, m-2) < l - 1 \leq q \text{ and } n > m - 2 \\ j_{l-2} & \text{for } \min(n, m-2) < l - 1 \leq q \text{ and } m - 2 > n. \end{cases}$$

3. Both $s_1 \dots s_n$ and $t_1 \dots t_m$ are tick permutations. The parameters are exactly as in case 1, except that $p_1 \dots p_q$ is a tick permutation.

The permutation π as constructed clearly has $\sigma, \tau \preccurlyeq \pi$ in each case and hence Z is atomic. □

2.5 Merges

We can regard merges as a generalisation of juxtapositions, in that juxtapositions are contained within merges of classes. They are partially motivated by considering permuting machines (for example stacks or queues) connected together in 'parallel': the input can be partitioned into two subsequences which are fed into 2 machines A and B, whose outputs are then merged [5].

Definition 2.5.1. Let σ, τ be permutations. The permutation π is a *merge* of σ and τ if it consists of two subsequences, one order isomorphic to σ, the other to τ. We shall write $\pi = \sigma \sqcup \tau$.

Example 2.5.2. 136542 is a merge of 123 and 321, in three different ways: (136, 542), (135, 642) and (134, 652).

We can extend our definition to sets of permutations in the obvious way. For sets X and Y,

$$X \sqcup Y = \{\sigma \sqcup \tau : \sigma \in X, \tau \in Y\}.$$

It is clear that if X and Y are closed classes then $X \sqcup Y$ is also closed.

Although the concept of a merge was motivated by connecting permuting machines in parallel, there are numerous cases where a machine, formed by connecting in parallel two machines with closed classes X and Y respectively, does not have the associated closed class $X \sqcup Y$:

permuting machines motivated this definition, not all such machines connected in parallel possess the property that the associated closed class is exactly the merge of their respective closed classes:

Example 2.5.3. We can connect in parallel the infinite queue (closed class $I = \mathcal{A}(21)$) with a stack (closed class $\mathcal{A}(231)$), but this machine cannot sort $632541 \in \mathcal{A}(21) \sqcup \mathcal{A}(231)$ (632541 can be written as a merge of $52143 \in \mathcal{A}(231)$ and $1 \in \mathcal{A}(21)$) [12].

A sufficient condition, for two machines connected in parallel to have closed class equal to the merge of their respective classes, is that both machines have the *waiting property*: the ability of each machine to hold an arbitrary number of tokens in its output prior to recombining it with the output from the other machine [12]. That it is not necessary can be seen by considering an infinite queue Q connected in parallel to a finite queue (Q_m, also with closed class I), which has closed class $\mathcal{A}(21) \sqcup \mathcal{A}(21) = \mathcal{A}(321)$ [12]. Permutations in $\mathcal{A}(321)$ are simply those which can be expressed as two interleaving increasing sequences (number 2 of Example 1.2.5).

Example 2.5.4. We demonstrate sorting the permutation 15263478 through $Q \sqcup Q_3$. Begin by passing symbol 1 directly through Q_3, and then symbol 5 enters Q and is stored. Symbol 2 can pass straight through Q_3, and symbol 6 enters Q. We now have 12 in the output, and 56 in Q. Symbols 3 and 4 can both be passed directly through Q_3, at which point we can pop symbols 5 and 6 from Q into the output, giving output 123456. Then 7 and 8 can be passed through either of Q or Q_3.

Finite Bases It is unknown whether the merge of two finitely based classes is always finitely based. We have one result [5]:

Proposition 2.5.5. *Let F_k denote the (finite) closed class of all permutations of length at most k and let X be any closed class. Then*

1. $X \sqcup F_{m+n} = X \sqcup F_m \sqcup F_n$,

2. *If X is finitely based then so is $X \sqcup F_n$ for all n.*

There are a few other cases where we do have finite bases, and we present two examples from [12]:

Example 2.5.6. (i) $I \sqcup R$ has basis elements made up exactly of the wreath products of 12 and 21 (see Section 2.8), namely $\{3412, 2143\}$.

(ii) $R \sqcup R = \mathcal{A}(12) \sqcup \mathcal{A}(12) = \mathcal{A}(123)$.

Atomicity The merge of two atomic classes does not need to be atomic, and Example 2.4.4 again provides a counterexample:

Example 2.5.7. Let $X = Y = \{\varnothing, 1\}$, the atomic class consisting of the empty permutation and the trivial permutation. Then

$$X \sqcup Y = \{\varnothing, 1, 12, 21\} \cong [X, Y]$$

which we have already seen is not atomic.

2.6 Differentiation

This construction is different from the others in that we are constructing a smaller closed class from a larger one. The results presented here are all from [12]:

Definition 2.6.1. If σ is a permutation, then the *derivative* of σ, $\partial\sigma$ is the permutation order isomorphic to σ with its first term removed. Similarly, for a set X of permutations the derivative is $\partial X = \{\partial\sigma : \sigma \in X\}$.

Proposition 2.6.2. *If X is a closed class then the set ∂X is also closed.*

Proof. We can regard any $\sigma \in \partial X$ as being the derivative of the permutation $s\sigma \in X$ for some symbol s. Then any $\tau \preccurlyeq \sigma$ can be regarded as the derivative of $t\tau \preccurlyeq s\sigma \in X$ for suitable t (which in the involvement we can certainly match to s), and since X is closed, $t\tau \in X$ implies $\tau \in \partial X$. $\square$

We do not, however, have the finite basis property for derivatives: X finitely based does not mean ∂X is finitely based. In fact, we can also have X infinitely based but ∂X finitely based. Examples of both are given in [12]. We do have the finite basis property for natural classes:

Proposition 2.6.3. *If X is a finitely based natural class then ∂X is also finitely based.*

Atomicity Derivatives play an important role in atomicity, and in particular for natural classes. First we present two results about atomicity, both of which are unproved but clear [12].

Lemma 2.6.4. *If $X = \mathrm{Sub}(\pi : A \to B)$ and if a is the first element in the domain A then $\partial X = \mathrm{Sub}(\pi : A \setminus \{a\} \to B)$ with π restricted to its new domain. If A does not have a first element then $\partial X = X$.*

Lemma 2.6.5. *If X can be written as a union of classes, $X = Y_1 \cup \ldots \cup Y_n$, say, then $\partial X = \partial Y_1 \cup \ldots \cup \partial Y_n$.*

Lemma 2.6.4 tells us that if X is atomic then so is ∂X. We can, however, have the derivative of a non-atomic class being atomic:

Example 2.6.6. Let $X = \{\varnothing, 1, 12, 21\}$. This is not atomic by Example 2.4.4, but the derivative $X = \{\varnothing, 1\}$ is clearly atomic.

Considerable attention has been given to natural classes in [12] and here we present just one such theorem.

Proposition 2.6.7. *A natural class $X = \mathrm{Sub}(\pi : \mathbb{N} \to \mathbb{N})$ is sum complete if and only if $X = \partial X$.*

There are several results related to the structure of our permutation π on the natural numbers, and also bounds on n minimal such that $\partial^n X = \partial^{n+1} X$. These results are outwith the main focus of this thesis, however, and the reader is referred to Chapter 5 of [12].

2.7 Profile Classes

Atkinson [4] defines the notion of the profile of a permutation, which leads to another form of construction. The notion of profile is also a special case of wreath products which we shall introduce next, and they are also central to the wreath product finite basis results in Chapter 3.

Definition 2.7.1. The permutation σ is said to have *profile* $\pi = [p_1, \ldots, p_m]$ if σ can be partitioned into segments $\sigma = \sigma_1 \ldots \sigma_m$ with m minimal, subject to

1. each σ_i is a non-empty sequence of increasing symbols,
2. $\sigma_i < \sigma_j$ if and only if $p_i < p_j$.

We will normally write σ^* to mean the profile of σ.

Example 2.7.2. The permutation 2346751 has profile 2431 because of the segments 234, 67, 5 and 1.

Not all permutations are profiles; a valid profile must contain no segment of the form $i, i+1$, i.e. it must be irreducible. Each permutation clearly has a unique profile because in the above definition we took m to be minimal. The converse is not true, however. In fact, from [4] we have

Lemma 2.7.3. *If π is a valid profile of length m then the number of permutations of length n with profile π is $\binom{n-1}{m-1}$.*

We may construct closed classes of permutations whose profiles all lie in some set of profiles. In general, a set of profiles cannot be closed as this would require involving non-irreducible elements (the only cases where this is possible are in R or $R_m = \{1, 21, \ldots, m \ldots 21\}$). Thus we define the following structure for profile classes:

Definition 2.7.4. A set P of permutations is *profile-closed* if all its members are valid profiles, and whenever $\mu \preccurlyeq \pi \in P$ is a valid profile then $\mu \in P$. The *profile-closure* of a set of profiles is the smallest profile-closed set containing it.

Example 2.7.5. The profile closure of $\{1324\}$ is $\{1324, 132, 213, 21, 1\}$. Note $123 \preccurlyeq 1324$ but 123 is not a valid profile.

This profile-closed structure is sufficient to create a closed class construction as given in [4]. Again the proof gives us details of the form of basis elements of such a class.

Theorem 2.7.6. *If P is a profile-closed set of permutations then $\mathcal{P}(P)$, the set of permutations whose profiles lie in P, is closed. Moreover, if P is finite then $\mathcal{P}(P)$ has a finite basis.*

Proof. Let $\sigma \in \mathcal{P}(P)$. From the definitions, if σ has profile σ^* and $\tau \preccurlyeq \sigma$ then τ has profile τ^* where $\tau^* \preccurlyeq \sigma^*$. Hence $\tau^* \in P$ and so $\tau \in \mathcal{P}(P)$. Thus $\mathcal{P}(P)$ is closed.

Let $\beta \in \mathcal{B}(\mathcal{P}(P))$ be a permutation on the symbols $1, \ldots, m$. If β has two adjacent consecutive symbols $i, i+1$ then β and $\beta - i$ have the same profile. However we must have $\beta - i \in \mathcal{P}(P)$ and so its profile lies in P. Thus β has a profile in P and so $\beta \in \mathcal{P}(P)$, a contradiction. Thus β is irreducible.

The permutation $\beta - m$ can therefore have at most two adjacent consecutive symbols (occurring only if the two symbols on either side of m in β were consecutive), and so it has length at most 1 more than the length of its profile. By minimality of β we must have $\beta - m \in \mathcal{P}(P)$ so its profile lies in P. As P is finite we have a bound on the length of the profile of $\beta - m$, and hence a bound on the length of β. $\square$

The proof of the above tells us that elements in the basis of a class $\mathcal{P}(P)$ must be irreducible. Classes with this property must be expanded by Lemma

1.3.19. We shall see in Section 2.8 that this result follows from properties of the wreath product and the set of identity permutations $I = \{1, 12, 123, \ldots\}$.

2.7.1 Skew and Strong Profiles

We will see that profiles play a key role in developing finite basis conditions on wreath products $X \wr I$. We can adapt these finite basis results to related wreath products (see Chapter 3), but in order to do this we first need to extend our definition of profile. Here we present two profiles both mentioned in [4], and both of which we shall use later.

Definition 2.7.7. The permutation σ is said to have *skew profile* $\pi = [p_1, \ldots, p_m]$ if σ can be partitioned into segments $\sigma = \sigma_1 \ldots \sigma_m$ with m minimal, subject to

1. each σ_i is a non-empty sequence of consecutive decreasing symbols,
2. $\sigma_i < \sigma_j$ if and only if $p_i < p_j$.

We will normally write $\sigma^{\downarrow}$ to mean the skew profile of σ.

Note that a valid skew profile must not contain any segment of the form $i+1, i$, and hence in the obvious sense must be 'skew irreducible'.

Example 2.7.8. The permutation 5476132 has skew profile 3412 because of the segments 54, 76, 1 and 32.

Definition 2.7.9. The permutation σ is said to have *strong profile* $\pi = [p_1, \ldots, p_m]$ if σ can be partitioned into segments $\sigma = \sigma_1 \ldots \sigma_m$ with m minimal, subject to

1. each σ_i is a non-empty sequence of consecutive decreasing or consecutive increasing symbols,
2. $\sigma_i < \sigma_j$ if and only if $p_i < p_j$.

We will normally write $\sigma^{\updownarrow}$ to mean the strong profile of σ.

Unlike the usual profile and the skew profile, the strong profile can have both segments of the form $i, i+1$ and $i+1, i$, and as such every permutation is a valid strong profile.

Example 2.7.10. The permutation 5763421 has strong profile 3421 because of the segments 5, 76, 34 and 21.

As for the normal profile σ^*, we can define the notions of *skew profile-closed* and *strong profile-closed* in the obvious way. Observe, however, that because there is no restriction on the structure of strong profiles, a strong profile-closed set is just the same as a closed set.

We can obtain similar results to Theorem 2.7.6 for both the skew and strong profiles.

Theorem 2.7.11. *If P is a skew profile-closed set of permutations then $\mathcal{P}^{\downarrow}(P)$, the set of permutations whose skew profiles lie in P, is closed. Moreover, if P is finite then $\mathcal{P}^{\downarrow}(P)$ has a finite basis.*

Proof. Analogous to the proof of 2.7.6. □

Theorem 2.7.12. *If P is a strong profile-closed set of permutations then $\mathcal{P}^{\updownarrow}(P)$, the set of permutations whose strong profiles lie in P, is closed. Moreover, if P is finite then $\mathcal{P}^{\updownarrow}(P)$ has a finite basis.*

Proof. We proceed as in 2.7.6. That $\mathcal{P}^{\updownarrow}(P)$ is closed is the same as before. We now observe that any $\beta \in \mathcal{B}(\mathcal{P}^{\updownarrow}(P))$ must have neither any consecutive increasing nor consecutive decreasing segments - they must all be strongly irreducible. We then argue as before to obtain the bound on the length of β. □

Remark 2.7.13. Once the concept of wreath product has been introduced, we shall see that this theorem is equivalent to showing that $F \wr (I \cup R)$ has finite basis for all finite closed classes F. We will, however, prove a stronger result later, and this result will no longer hold significance.

2.8 The Wreath Product

The wreath product of two closed classes is somewhat more complicated in terms of structure than the constructions we have looked at so far. Very little is known in general about the structure of basis elements - or indeed conditions for a wreath product class to have a finite basis. Most of what is known is from [8], and here we present a few of these results.

Definition 2.8.1. The *wreath product* of two not necessarily closed sets X and Y is the set $X \wr Y$ of permutations $\sigma = \alpha_1\alpha_2\dots\alpha_k$ such that

1. each α_i is a rearrangement of an interval,
2. each α_i is order isomorphic to a permutation of Y,
3. if for every i we pick a symbol a_i from α_i, then $a_1a_2\dots a_k$ is order isomorphic to a permutation in X (our particular choice of a_i does not matter because of condition 1).

An easier way to picture the wreath product is that we take an element $[a_1, a_2, \dots, a_k] \in X$ and replace each a_i with a permutation from Y, while still maintaining the 'block structure' set out by the element of X.

Example 2.8.2. The wreath product $X \wr Y$ of the singleton set $X = \{12\}$ with the set $Y = \{12, 21\}$ is the set $\{1234, 1243, 2134, 2143\}$. The wreath product $Y \wr X = \{1234, 3412\}$ demonstrates that we do not in general have $X \wr Y = Y \wr X$.

The following two results are trivial.

Lemma 2.8.3. *If X and Y are closed then $X \wr Y$ is closed.*

Lemma 2.8.4. *The wreath product is associative: $(X \wr Y) \wr Z = X \wr (Y \wr Z)$*

Elements of $X \wr Y$ may not necessarily have a unique representation of this form, but we can obtain conditions about extremal values. Let $\gamma \in X \wr Y$ be represented as $[a_1, \dots, a_k] \wr (\alpha_1, \dots, \alpha_k)$ where $a_1 \dots a_k \in X$ and all $\alpha_i \in Y$.

(I) k maximal. Then, for each i one of the following holds

(a) α_i is strongly indecomposable,

(b) $\alpha_i = \beta \oplus \gamma$ but the positive expansion of $a_1, \ldots, a_k$ at a_i is not in X,

(c) $\alpha_i = \beta \ominus \gamma$ but the negative expansion of $a_1, \ldots, a_k$ at a_i is not in X.

(II) k minimal. Then, for each $i < k$ we have one of the following

(a) $a_i \neq a_{i+1} \pm 1$,

(b) $a_i = a_{i+1} + 1$ but we have $\alpha_i \ominus \alpha_{i+1} \notin Y$,

(c) $a_i = a_{i+1} - 1$ but we have $\alpha_i \oplus \alpha_{i+1} \notin Y$.

The cases where $X = I$ or $X = S$, $Y = I$ or $Y = S$ (S the set of separable permutations: see number 3 of Example 1.2.5) cause the above to simplify and we obtain the following results. [8]

Lemma 2.8.5. *For any closed set X we have*

1. *$X \wr I$ is the expansion of X,*

2. *$X \wr S$ is the strong expansion of X,*

3. *$I \wr X$ is the completion of X,*

4. *$S \wr X$ is the strong completion of X.*

Remark 2.8.6. In relation to the previous section on profile-closed classes P, we now observe that the construction $\mathcal{P}(P)$ is simply the wreath product $P \wr I$, which is why this construction gives us expanded classes. Of course the (profile-closed) set P is not in general closed, so we still need to check that the wreath product $P \wr I$ is closed, but this follows fairly trivially.

Remark 2.8.7. From Lemma 1.3.19 we know that basis elements of the above 4 wreath products must be irreducible, strongly irreducible, indecomposable and strongly indecomposable respectively.

2.8.1 Wreath Closure

We already have the concepts of closure, completion and expansion of sets and closed classes. It is therefore natural to consider an analogous property that closed classes might have with respect to the wreath product, namely when a closed class is equal to the wreath product with itself. Some of this has been covered in [12].

Definition 2.8.8. A set of permutations X is *wreath closed* if X is closed and obeys $X = X \wr X$.

For any closed class X, the *wreath closure* is the smallest wreath closed class containing X, and will be denoted $\mathcal{W}(X)$.

Equivalently, we can say a closed class X is wreath closed if every wreath product of elements in X results in another element of X. We see this is equivalent because for any closed set X we have $X \subseteq X \wr X$.

Example 2.8.9. We can immediately think of two easy classes that possess this property: the set I of identity permutations, and the set R of reverse permutations.

We are aiming to determine what form the basis elements of wreath closed classes must take, and hence also what basis elements we have when considering the wreath closure of arbitrary closed classes.

We first note that, in a wreath closed class X, the basis of X must equal the basis of $X \wr X$, hence any $\beta \in \mathcal{B}(X)$ cannot be expressible as a wreath product of elements in X. This gives rise to the following definition:

Definition 2.8.10. A permutation σ is said to be *wreath indecomposable* if whenever σ is expressed as a wreath product, $\sigma = s_1 \dots s_k \wr (\sigma_1, \dots, \sigma_k)$, then either $k = 1$ or all of $\sigma_1, \ldots, \sigma_k$ are equal to the trivial permutation.

Example 2.8.11. The wreath indecomposable permutations of length 4 or 5 are [2]

- 2413, 3142;
- 25314, 35142, 31524, 42513, 24153, 41352.

Note that there are no wreath indecomposable permutations of length 3. However, both 12 and 21 are wreath indecomposable.

It is not difficult to see that this is the same condition as we imposed for a permutation to be simple (see Section 1.3). This also tells us that a wreath indecomposable permutation of length ≥ 3 must be both strongly irreducible and strongly indecomposable.

The following theorem is now fairly easily obtained.

Theorem 2.8.12. *X is a wreath closed class if and only if every basis element is simple.*

Proof. ($\Rightarrow$) Let X be wreath closed with $\beta \in \mathcal{B}(X)$ and suppose β is not simple. Then we can express β as a wreath product

$$\beta = [b_1, \ldots, b_k] \wr (\beta_1, \ldots, \beta_k).$$

Since β is a basis element and is therefore minimally not in X (with respect to involvement), all of $[b_1, \ldots, b_k]$ and $\beta_1, \ldots, \beta_k$ lie in X. Since X is wreath closed this tells us that β must lie in X, a contradiction.

($\Leftarrow$) If every basis element is simple, then suppose there exists a permutation $[s_1, \ldots, s_k]$ and permutations $\sigma_1, \ldots \sigma_k$ all in X such that $\alpha = [s_1, \ldots, s_k] \wr (\sigma_1, \ldots \sigma_k) \notin X$.

Then there exists a basis element β of X such that $\beta \preccurlyeq \alpha$. We now observe that, since β is wreath indecomposable, the only way β can be involved in α is if:

1. β has at most one element from each of $\sigma_1, \ldots, \sigma_k$, and hence $\beta \preccurlyeq [s_1, \ldots, s_k]$, which is impossible,
2. β is entirely contained within a single block σ_j (for some j). But then $\beta \in X$, which is impossible.

Thus we cannot have $\beta \preccurlyeq \alpha$ for any basis element β and so $\alpha \in X$. $\square$

Example 2.8.13. The class S of separable permutations has basis $\mathcal{B}(S) = \{2413, 3142\}$ and so is wreath closed.

From this, we can immediately deduce some properties about basis elements of the wreath closure of any closed class X [12].

Proposition 2.8.14. *Let X be a closed class. Then $\beta \in B = \mathcal{B}(\mathcal{W}(X))$ if and only if β is minimal (under involvement) subject to*

(i) β is simple,

(ii) $\beta \notin X$.

Proof. Our key observation is that any permutation in $\mathcal{W}(X)$ is either in X or is not simple. For suppose there exists a simple permutation $\sigma \in \mathcal{W}(X) \backslash X$, then the class defined by $\mathcal{A}(B \cup \{\sigma\})$ contains X and, as its basis elements are all simple, it is wreath closed by Theorem 2.8.12, contradicting minimality of $\mathcal{W}(X)$.

If $\beta \in B$ then β is simple by Theorem 2.8.12 and $\beta \notin X$ since $X \subseteq \mathcal{W}(X)$. Furthermore, β is minimal since if $\beta_1 \prec \beta$ then $\beta_1 \in \mathcal{W}(X)$ and not both of (i) and (ii) can hold.

Conversely, if β is not a basis element of $\mathcal{W}(X)$ then either $\beta \in \mathcal{W}(X)$, in which case not both of (i) and (ii) can hold, or β properly involves a basis element for which (i) and (ii) do hold, in which case β would not be minimal. $\square$

How we form the wreath closure is less obvious. We know that all its basis elements must be simple, and hence they must all be both strongly irreducible and strongly indecomposable. Thus the wreath closure of a set must be both strongly expanded and strongly complete by Lemma 1.3.19. By a double application of Lemma 2.8.5 we then know that $S \wr X \wr S \subseteq \mathcal{W}(X)$. Furthermore, our proof of the above Proposition makes the observation that the only simple elements of $\mathcal{W}(X)$ are those in X itself, but beyond that the wreath closure is unknown.

2.8.2 A Wreath Product with the Finite Basis Property

Atkinson and Stitt [8] used the result that $X \wr I$ is the expansion of X to obtain a finite basis condition. The first result we require is the format that such basis elements must take.

Lemma 2.8.15. *(1) $\beta \in \mathcal{B}(X \wr I)$ if and only if β is minimal (under involvement) subject to*

(i) β is irreducible,

(ii) $\beta \notin X$.

(2) $\beta \in \mathcal{B}(I \wr X)$ if and only if β is minimal (under involvement) subject to

(i) β is indecomposable,

(ii) $\beta \notin X$.

Proof. For part (1) we first note that any permutation in $X \wr I$ either belongs to X or is not irreducible. Suppose $\beta \in \mathcal{B}(X \wr I)$. Then since $X \wr I$ is expanded by Lemma 2.8.5, β must be irreducible by Lemma 1.3.19. Moreover, $\beta \notin X$ because $X \subseteq X \wr I$ and $\beta \notin X \wr I$. β is minimal since if $\beta_1 \prec \beta$ then we would have $\beta_1 \in X \wr I$ and not both of (i) and (ii) can hold for β_1.

Conversely, if β is not a basis element of $X \wr I$ then either $\beta \in X \wr I$, in which case not both of (i) and (ii) can hold, or β properly involves a basis element for which (i) and (ii) do hold, in which case β would not be minimal.

The proof for part (2) is similar. The key observation is that any permutation in $I \wr X$ either belongs to X or is not indecomposable. We must also note that $I \wr X$ is complete. $\square$

The second observation we require is a bound on the length of irreducible permutations containing a given permutation. We include the proof as it will be adapted in the next chapter to provide bounds on the length of other wreath products.

Lemma 2.8.16. *Let β be any permutation and σ a permutation minimal subject to*

(i) $\beta \preccurlyeq \sigma$,

(ii) σ is irreducible.

Then $|\sigma| \leq 2|\beta| - 1$.

Proof. Let $\beta = b_1 \dots b_k$ and $\sigma = s_1 \dots s_n$ where $s_{i_1}, \dots, s_{i_k}$ is the lexicographically leftmost subsequence of σ order isomorphic to β. We now take another subsequence σ' of σ which comprises the symbols $s_{i_1}, \dots, s_{i_k}$ together with extra s_i determined by the nature of the pairs (b_j, b_{j+1}). We construct σ' to possess no consecutive increasing entries.

If b_j, b_{j+1} is not consecutive increasing then observe that the symbols $s_{i_j}, s_{i_{j+1}}$ will also not be consecutive increasing.

Now suppose that b_j, b_{j+1} is consecutive increasing. If there were a symbol s_q with $i_j < q < i_{j+1}$ and $s_{i_j} < s_q < s_{i_{j+1}}$ then the symbol b_{j+1} could have been matched to s_q rather than $s_{i_{j+1}}$, contradicting our choice of $s_{i_1}, \dots, s_{i_k}$ as the lexicographically leftmost sequence of σ order isomorphic to β.

Thus any such s_q must obey $s_q < s_{i_j}$ or $s_q > s_{i_{j+1}}$, and in either case we include s_q in σ'. Note that for each pair we need only add one such s_q to σ'

to ensure no consecutive increasing sequences, and hence s_{i_j} and $s_{i_{j+1}}$ will correspond to distinct symbols in the profile of σ'.

Otherwise, s_{i_j} and $s_{i_{j+1}}$ are adjacent in σ. By the irreducibility of σ, there exists an $s_r \in \sigma$ such that $r < i_j$ or $r > i_{j+1}$ and $s_{i_j} < s_r < s_{i_{j+1}}$, and we include this symbol in σ', ensuring s_{i_j} and $s_{i_{j+1}}$ will correspond to distinct symbols in the profile of σ'.

So now we have constructed σ' so that its profile, σ^*, involves β. But since $\sigma^* \preccurlyeq \sigma$ and σ was minimal, we must have $\sigma = \sigma^* = \sigma'$. Our construction for σ' however tells us that $|\sigma'| = k + l$ where l is the number of symbols we have had to add due to the pairs b_j, b_{j+1} being consecutive increasing. In the worst case, there are at most $k-1$ pairs which are consecutive increasing, completing the proof. $\square$

Remark 2.8.17. The important property of the above lemma is not just that we have formed σ' so that $\beta \preccurlyeq \sigma' \preccurlyeq \sigma$, but also the chain $\beta \preccurlyeq \sigma^* \preccurlyeq \sigma' \preccurlyeq \sigma$ which includes the profile σ^* of σ'. Here we have found that they are all equal, but when we adapt this proof for other finite basis results we will need to use other types of profile for which this will not happen.

We now have enough tools to prove the finite basis theorem from [8].

Theorem 2.8.18. *If X is a finitely based closed set then $X \wr I$ is also finitely based.*

Proof. Let $\sigma \in \mathcal{B}(X \wr I)$. Then σ is irreducible by Lemma 1.3.19 and because $X \wr I$ is expanded. Since $\sigma \notin X$ there exists $\beta \in \mathcal{B}(X)$ such that $\beta \preccurlyeq \sigma$. We choose a permutation σ' minimal such that

1. $\beta \preccurlyeq \sigma' \preccurlyeq \sigma$,

2. σ' irreducible,

but then by Lemma 2.8.16, $|\sigma'|$ is bounded in terms of $|\beta|$. σ' however must lie in $\mathcal{B}(X \wr I)$ by Lemma 2.8.15 and so $\sigma = \sigma'$ and thus we have a bound on σ. $\square$

Example 2.8.19. Consider the set of stack sortable permutations $X = \mathcal{A}(231)$. Then $X \wr I$ is finitely based, and by Lemma 2.8.16 its basis elements can have length at most 5. A case by case search on those permutations that involve 231 (these are the only ones that could lie in the basis of $X \wr I$) gives the basis as $\{2431, 3241, 2413, 3142\}$ [8].

The reverse wreath product, $I \wr X$ (the sum completion), is not necessarily finitely based, as the following shows [8].

Theorem 2.8.20. *Let $X = \mathcal{A}(321654)$. Then $I \wr X$ is not finitely based.*

Observe this is the sum completion of $X = \mathcal{A}(321654)$, and we saw in Example 2.3.1 that $X \oplus X$ is infinitely based. The same infinite antichain works here.

Proof. Consider the following set of permutations, defined for $m > 2$:

$$\begin{aligned}\beta_m &= 3, 2, 5, 1, 7, 4, 9, 6, 11, 8, \ldots, 2i-1, 2i-4, 2i+1, 2i-2, \ldots \\ &\quad\; 2m-1, 2m-4, 2m+2, 2m-2, 2m+1, 2m.\end{aligned}$$

Excluding the first and last four symbols, β_m is made up of interleaving odd-valued and even-valued symbols, as indicated by the typical segment $2i-1, 2i-4, 2i+1, 2i-2$. Moreover, the segment 3251 overlaps 5174, which overlaps 7496, etc., and so we note that β_m is indecomposable.

The sequences $3, 2, 1$ and $2m+2, 2m+1, 2m$ are the only decreasing subsequences of length 3 and therefore $3, 2, 1, 2m+2, 2m+1, 2m$ is the unique subsequence order isomorphic to 321654. In particular, $\beta_m \notin X$ for all $m > 2$.

We claim that β_m satisfies condition (2) of Lemma 2.8.15, and hence $\beta_m \in \mathcal{B}(I \wr X)$. For this we need to show β_m is minimally not in $I \wr X$.

Consider the effect of removing any of the symbols from β_m. If we omit any of $3, 2, 1, 2m+2, 2m+1$ or $2m$ then 321654 is no longer involved and so the result lies in X, violating condition (ii) of 2.8.15. If we omit any other symbol, then what results is sum decomposable, violating condition (i) of 2.8.15. Hence $\beta_m \in \mathcal{B}(I \wr X)$, and thus $I \wr X$ has infinite basis. □

This theorem alone tells us that looking for generalised finite basis properties of wreath products is non-trivial and we certainly do not have X, Y finitely based implying $X \wr Y$ finitely based.

2.8.3 Atomicity

As stated in [12], the wreath product of two atomic classes is again atomic. We provide a proof below, noting that the construction of an element that obeys the Join Property in a wreath product is somewhat less obvious than the construction we used when considering direct and skew sums.

Proposition 2.8.21. *If X and Y are atomic classes then so is $X \wr Y$.*

Proof. Again we use the Join Property of atomic classes (condition 3 of Theorem 1.4.2). Pick any two elements $\lambda, \mu \in X \wr Y$. Then we can write

$$\begin{aligned} \lambda &= s_1 \dots s_k \wr (\sigma_1, \dots, \sigma_k) \\ \mu &= t_1 \dots t_l \wr (\tau_1, \dots, \tau_l) \end{aligned}$$

where $s_1 \dots s_k, t_1 \dots t_l \in X$ and $\sigma_1, \dots, \sigma_k, \tau_1, \dots, \tau_l \in Y$. Since X is atomic, there exists a permutation $a_1 \dots a_m \in X$ such that both $s_1 \dots s_k$ and $t_1 \dots t_l$ are involved in $a_1 \dots a_m$. We fix $a_{i_1}, \dots, a_{i_k}$ and $a_{j_1}, \dots, a_{j_l}$ to be subsequences of $a_1 \dots a_m$ that are order isomorphic to $s_1 \dots s_k$ and $t_1 \dots t_l$ respectively (such a choice may not be unique).

Since Y is atomic, for every pair of permutations σ_i and τ_j we have a $\beta_{ij} \in Y$ such that $\sigma_i, \tau_j \preccurlyeq \beta_{ij}$.

We now construct an element of $X \wr Y$, $\nu = a_1 \dots a_m \wr (\alpha_1, \dots, \alpha_m)$, which involves both λ and μ. For all $n = 1, \dots, m$ we pick α_n according to

- $\alpha_n = \sigma_p$ if $a_n = a_{i_p}$ but none of $a_{j_1}, \dots, a_{j_l}$ are equal to a_n,
- $\alpha_n = \tau_q$ if $a_n = a_{j_q}$ but none of $a_{i_1}, \dots, a_{i_k}$ are equal to a_n,
- $\alpha_n = \beta_{pq}$ if $a_n = a_{i_p} = a_{j_q}$,
- $\alpha_n = 1$ otherwise (that is, a_n does not occur in either $a_{i_1}, \dots, a_{i_k}$ or $a_{j_1}, \dots, a_{j_l}$).

By construction we have $\nu \in X \wr Y$ and also $\lambda, \mu \preccurlyeq \nu$. Hence $X \wr Y$ is atomic. $\square$

Given this result, one may ask what conditions we have to place on X and Y to ensure $X \wr Y$ is atomic. If X is not atomic, then we may write it as a union $X = X_1 \cup X_2$ of smaller classes, neither of which is contained in the other. As discussed in [12], we do not know much about the two classes X_1 and X_2 – they certainly do not need to be atomic.

However, writing $X = X_1 \cup X_2$ allows us to express our wreath product $X \wr Y$ as $(X_1 \cup X_2) \wr Y = (X_1 \wr Y) \cup (X_2 \wr Y)$. Here we have expressed our wreath product as a union of two closed classes. So if X is not atomic then $X \wr Y$ cannot be atomic for any Y unless $X_1 \wr Y = X_2 \wr Y$. The exact conditions where this exception may or may not occur are unknown.

Whether $X \wr Y$ can be atomic if Y is not atomic is also unknown.

Chapter 3

Finitely Based Wreath Products

In this chapter we continue to investigate the properties of the wreath product as defined in the last chapter. We have already demonstrated the finite basis property for $X \wr I$, and here we offer a new but related proof which is then adapted to other previously unknown wreath products.

3.1 Profiles and the Finite Basis Property

The proof of Lemma 2.8.16 is central to achieving the result that $X \wr I$ is finitely based if X is finitely based. As mentioned in remarks in the previous chapter, we are interested in the profile σ^* of $\sigma \in \mathcal{B}(X \wr I)$, and the fact that it involves a basis element of X. We can therefore formulate a new proof that discusses elements in terms of their profiles.

We may write any permutation σ uniquely as a wreath product of its profile with elements of the class I of identity permutations:

$$\sigma = \sigma^* \wr (\iota_1, \ldots, \iota_k)$$

where $\iota_i \in I$ for all i and $|\sigma^*| = k$. The uniqueness comes about as a result of picking k to be minimal, and hence each ι_i is maximal. As such, we can say that the ι_i partition σ into maximal segments. Two such maximal segments cannot together be consecutive increasing, for suppose ι_i and ι_{i+1}

are consecutive increasing, then we could take the larger segment made up of both of them, contradicting our maximality. We then make the following observation:

Lemma 3.1.1. *$\sigma \in X \wr I$ if and only if $\sigma^* \in X$.*

Proof. ($\Leftarrow$) Trivial as $\sigma = \sigma^* \wr (\iota_1, \dots, \iota_k)$ with each $\iota_i \in I$.

($\Rightarrow$) We decompose $\sigma = \sigma_1 \dots \sigma_k$ into maximal consecutive increasing segments, and write the profile $\sigma^* = s_1 \dots s_k$, with $\sigma_i < \sigma_j \Leftrightarrow s_i < s_j$.

Now suppose we have $\tau = t_1 \dots t_l \in X$ such that $\sigma = \tau \wr (\iota_1, \dots, \iota_l)$ for some $\iota_1, \dots, \iota_l \in I$. Then we can partition σ in terms of τ as $\sigma = \tau_1 \dots \tau_l$ with $\tau_i < \tau_j \Leftrightarrow t_i < t_j$.

Since our original partition $\sigma_1 \dots \sigma_k$ had k minimal, we know $l \geq k$. We now claim that we can partition each σ_j into segments from $\tau_1, \dots, \tau_l$. For if there exists a τ_i that overlaps two segments σ_j and σ_{j+1}, then since τ_i is consecutive increasing we must have $\sigma_j \sigma_{j+1}$ making up a consecutive increasing segment, contradicting the maximality of $|\sigma_j|$ and $|\sigma_{j+1}|$. Thus we choose $i_1, i_2, \dots, i_k = l$ such that

$$\begin{array}{rcl} \sigma_1 & = & \tau_1 \dots \tau_{i_1} \\ \sigma_2 & = & \tau_{i_1+1} \dots \tau_{i_2} \\ & \vdots & \\ \sigma_k & = & \tau_{i_{k-1}+1} \dots \tau_l. \end{array}$$

Now we construct the mapping $s_j \mapsto t_{i_j}$ which is order-preserving:

$$\begin{array}{rcl} s_p < s_q & \Longleftrightarrow & \sigma_p < \sigma_q \\ & \Longleftrightarrow & \tau_{i_p} < \tau_{i_q} \text{ since } \tau_{i_p} \preccurlyeq \sigma_p \text{ and } \tau_{i_q} \preccurlyeq \sigma_q \\ & \Longleftrightarrow & t_{i_p} < t_{i_q} \end{array}$$

and so the profile $\sigma^* \preccurlyeq \tau \in X$, and by the closure of X we have $\sigma^* \in X$. $\square$

As this is a two-way implication we also obtain the negative statement: $\sigma \notin X \wr I$ if and only if $\sigma^* \notin X$. Note that this does not give us any information on whether σ or σ^* lie in their respective bases – it will in general not be true that $\sigma \in \mathcal{B}(X)$ if and only if $\sigma^* \in \mathcal{B}(X \wr I)$. Note however that we know $X \wr I$ is expanded, and hence all basis elements are irreducible. An element σ that is irreducible has the property $\sigma = \sigma^*$, but we do not actually need this result to prove that $X \wr I$ has a finite basis: the fact that $\sigma^* \preccurlyeq \sigma$ is sufficient, and true for any permutation.

The following technical lemma is our equivalent to Lemma 2.8.16, and its proof is almost identical. The only difference is that we have stipulated that π must be involved in σ, but this is exactly what we set up during the proof of Lemma 2.8.16.

Lemma 3.1.2. *Let σ be any permutation and $\tau \preccurlyeq \sigma^*$. Then if π is minimal subject to*

$$\tau \preccurlyeq \pi^* \preccurlyeq \pi \preccurlyeq \sigma$$

we have $|\pi| \leq 2|\tau| - 1$.

Proof. Let $\tau = t_1 \dots t_k$ and $\sigma = s_1 \dots s_n$ where $s_{i_1}, \dots, s_{i_k}$ is the lexicographically leftmost subsequence of σ order isomorphic to τ. We now take another subsequence π of σ which comprises the symbols $s_{i_1}, \dots, s_{i_k}$ together with extra s_i determined by the nature of the pairs (t_j, t_{j+1}). We construct π to possess no consecutive increasing entries.

If t_j, t_{j+1} is not consecutive increasing then observe that the symbols $s_{i_j}, s_{i_{j+1}}$ will also not be consecutive increasing.

Now suppose that t_j, t_{j+1} is consecutive increasing. If there were a symbol s_q with $i_j < q < i_{j+1}$ and $s_{i_j} < s_q < s_{i_{j+1}}$ then the symbol t_{j+1} could have been matched to s_q rather than $s_{i_{j+1}}$, contradicting our choice of $s_{i_1}, \dots, s_{i_k}$ as the lexicographically leftmost subsequence of σ order isomorphic to τ.

Thus any such s_q must obey $s_q < s_{i_j}$ or $s_q > s_{i_{j+1}}$, and in either case we include s_q in π. Note that for each pair we need only add one such s_q to π to ensure no consecutive increasing sequences, and hence s_{i_j} and $s_{i_{j+1}}$ will correspond to distinct symbols in the profile of π.

Otherwise we must have $s_{i_j}, s_{i_{j+1}}$ adjacent in σ. Since σ is irreducible they cannot be consecutive increasing, and so there exists a symbol s_r in σ with $r < i_j$ or $r > i_{j+1}$ and $s_{i_j} < s_r < s_{i_{j+1}}$. We place one such r into π, thus again ensuring s_{i_j} and $s_{i_{j+1}}$ will correspond to distinct symbols in the profile of π.

So now we have constructed π so that its profile, π^*, involves τ. Moreover, our construction of π comprises the k symbols $s_{i_1}, \ldots, s_{i_k}$ together with at most one extra symbol per consecutive pair - that is a further $k-1$, and hence we obtain the required bound. □

Theorem 3.1.3. *If X is a finitely based closed class then so is $X \wr I$.*

Proof. Let $\sigma \in \mathcal{B}(X \wr I)$. Then $\sigma^* \notin X$ and so there exists a $\beta \in \mathcal{B}(X)$ such that $\beta \preccurlyeq \sigma^*$. We now construct a π minimal such that

$$\beta \preccurlyeq \pi^* \preccurlyeq \pi \preccurlyeq \sigma$$

which we may do by Lemma 3.1.2. Moreover $|\pi|$ is bounded in terms of $|\beta|$. Now we have $\pi^* \succcurlyeq \beta \in \mathcal{B}(X)$, and so by Lemma 3.1.1, $\pi \notin X \wr I$. But $\pi \preccurlyeq \sigma \in \mathcal{B}(X \wr I)$ and so we must have $\pi = \sigma$, and hence $|\sigma|$ is bounded in terms of $|\beta|$. □

3.2 Finite Basis Property of $X \wr I_m$

Here we consider the wreath product of a finitely based class X with I_m, the set of identity permutations of length at most m, that is

$$I_m = \{1, 12, 123, \ldots, 12\ldots m\}.$$

I_m is finitely based and it can easily be seen that $\mathcal{B}(I_m) = \{21, 12\ldots m+1\}$.

For what follows we will assume $m \geq 2$ as the case $m = 1$ gives us the trivial set $T = \{1\}$. Notice that $X \wr T = X$, and hence if X is finitely based then $X \wr T$ must also be finitely based.

Elements of $X \wr I_m$ take the form of elements in X with each symbol of these elements expanded into increasing sequences of length at most m. Note that if, for example, we have the element $12 \in X$, then we will have the element $[1, 2, \ldots, 2m] \in X \wr I_m$, and if $123 \in X$ then $X \wr I_m$ contains the identity element of length $3m$ and so on. Thus we may not necessarily form any kind of bound on the length of the maximal consecutive increasing sequence, unless X satisfies certain extra conditions. Using this approach therefore we can form little idea of the structure of basis elements of $X \wr I_m$ (whereas with $X \wr I$ we knew basis elements had to be irreducible), and so this is where we turn once again to the method developed in the previous section.

Our result for the basis of $X \wr I_m$ follows an identical approach to our proof of $X \wr I$. However, before we can embark on such a proof we need to define an analogue to the profile σ^* of a permutation σ.

Definition 3.2.1. Let σ be any permutation. Then the *m-profile*, $\sigma^{(m)} = s_1 s_2 \ldots s_k$, of σ is the permutation with k minimal, subject to

$$\sigma = \sigma^{(m)} \wr (\iota_1, \iota_2, \ldots, \iota_k)$$

with each $\iota_j \in I_m$.

Where $\sigma^{(m)}$ has two adjacent elements s_i, s_{i+1} such that $s_{i+1} = s_i + 1$ then we take $\iota_i = 12 \ldots m$, the maximal possible permutation in I_m. We do not strictly require this to define the m-profile, but in what follows we need to ensure our choices of ι_j are unique.

We can therefore write any element σ of $X \wr I_m$ as a unique wreath product $\sigma = \sigma^{(m)} \wr (\iota_1, \ldots, \iota_k)$ with each ι_j as defined above. Our analogue to Lemma 3.1.1 is therefore

Lemma 3.2.2. *$\sigma \in X \wr I_m$ if and only if $\sigma^{(m)} \in X$.*

The proof proceeds along the same lines as Lemma 3.1.1, although the forwards implication requires a little more thought as the wreath product with the m-profile may not be unique, and so we may encounter 'overlaps' which were impossible in the case $X \wr I$.

Proof. ($\Leftarrow$) Trivial as $\sigma = \sigma^{(m)} \wr (\iota_1, \ldots, \iota_k)$ with each $\iota_j \in I_m$.

($\Rightarrow$) We decompose $\sigma = \sigma_1 \ldots \sigma_k$ into maximal consecutive increasing segments as defined by the m-profile, and write the m-profile $\sigma^{(m)} = s_1 \ldots s_k$, with $\sigma_i < \sigma_j \Leftrightarrow s_i < s_j$. Moreover where $s_{i+1} = s_i + 1$ then $|\sigma_i| = m$.

Now suppose we have $\tau = t_1 \ldots t_l \in X$ such that $\sigma = \tau \wr (\iota_1, \ldots, \iota_l)$ for some $\iota_1, \ldots, \iota_l \in I_m$. Then we can partition σ in terms of τ as $\sigma = \tau_1 \ldots \tau_l$ with $\tau_i < \tau_j \Leftrightarrow t_i < t_j$ for all i, j, and $|\tau_i| \leq m$ for all i.

Since our original partition $\sigma_1 \ldots \sigma_k$ had k minimal, we know $l \geq k$. We now consider superimposing the decomposition $\tau_1 \ldots \tau_l$ over the decomposition $\sigma_1 \ldots \sigma_k$. We claim that we can partition each σ_j into segments or parts of segments from $\tau_1, \ldots, \tau_l$, and in particular that for each σ_j there exists a τ_{i_j} whose right hand end lies within σ_j.

Thus for each j we fix i_j so that τ_{i_j} is the rightmost segment from $\tau_1 \ldots \tau_l$ whose right hand end lies in σ_j. This exists for every j as otherwise we would have $\sigma_j \prec \tau_i$ for some i. Observe that this means that $\sigma_j \sigma_{j+1}$ is then a consecutive increasing segment and so $|\sigma_j| = m$. In this case $|\tau_i| > |\sigma_j| = m$, a contradiction.

To the right of τ_{i_j} in σ_j, we may have part of τ_{i_j+1}, and we will denote this part by $\mathcal{L}(\tau_{i_j+1})$. This left-hand end of τ_{i_j+1} cannot be all of τ_{i_j+1} by our choice of i_j, but it could be none of it (e.g. whenever $s_{j+1} \neq s_j + 1$).

At the beginning of each σ_j we will of course have the right-hand end of the permutation $\tau_{i_{j-1}+1}$, denoted $\mathcal{R}(\tau_{i_{j-1}+1})$. This will be determined by the decomposition of σ_{j-1}, and hence could be all or some of $\tau_{i_{j-1}+1}$, but not all.

Thus we can express σ_j as a decomposition into the τ_i and parts thereof, and in each case may be written

$$\sigma_j = \mathcal{R}(\tau_{i_{j-1}+1})\tau_{i_{j-1}+2} \dots \tau_{i_j}\mathcal{L}(\tau_{i_j+1}).$$

Now we can construct the order-preserving mapping $s_j \mapsto t_{i_j}$. For all $p, q = 1, \dots, k$:

$$\begin{aligned} s_p < s_q &\iff \sigma_p < \sigma_q \\ &\iff \tau_{i_p} < \tau_{i_q} \text{ since } \mathcal{R}(\tau_{i_p}) \preccurlyeq \sigma_p \text{ and } \mathcal{R}(\tau_{i_q}) \preccurlyeq \sigma_q \\ &\iff t_{i_p} < t_{i_q} \end{aligned}$$

and so the m-profile $\sigma^{(m)} \preccurlyeq \tau \in X$, and by the closure of X we have $\sigma^{(m)} \in X$. □

The other component we require is an equivalent of the bound given in Lemma 3.1.2. This is a slightly less straightforward construction as we can have consecutive increasing sequences occurring in our m-profile of σ. These we must 'expand' so that when we take the m-profile of our new element π these two consecutive increasing symbols lie in different blocks. This results in a larger bound than we obtained previously.

Lemma 3.2.3. *Let σ be any permutation and $\tau \preccurlyeq \sigma^{(m)}$. Then if π is minimal subject to*

$$\tau \preccurlyeq \pi^{(m)} \preccurlyeq \pi \preccurlyeq \sigma$$

we have $|\pi| \leq m(|\tau| - 1) + 1$.

Proof. Let $\tau = t_1 \dots t_l$, $\sigma^{(m)} = s_1 \dots s_k$ and write $\sigma = \sigma^{(m)} \wr (\iota_1, \dots, \iota_k)$ with each $\iota_j \in I_m$ and where s_i, s_{i+1} are consecutive increasing we arrange that $\iota_i = [1, 2, \dots, m]$. Also, let us suppose that $s_{i_1}, \dots, s_{i_l}$ is the lexicographically leftmost occurrence of τ in $\sigma^{(m)}$.

We construct a new permutation π as follows. π comprises the subsequence $s_{i_1}, \dots, s_{i_l}$ of $\sigma^{(m)}$, together with extra s_i determined by the nature of the pairs (t_j, t_{j+1}). These extra elements ensure the m-profile, $\pi^{(m)}$, still involves the permutation τ.

If t_j, t_{j+1} are not consecutive increasing then we do not need to add any further symbols to π as the symbols $s_{i_j}, s_{i_{j+1}}$ will also not be consecutive increasing. So suppose t_j, t_{j+1} are consecutive increasing. There are 3 possible cases:

1. $s_{i_j}, s_{i_{j+1}}$ are not adjacent in $\sigma^{(m)}$, so there exists a symbol s_q such that $i_j < q < i_{j+1}$. If this symbol were such that $s_{i_j} < s_q < s_{i_{j+1}}$ then we could have matched t_{j+1} to s_q rather than $s_{i_{j+1}}$, contradicting our leftmost choice of $s_{i_1}, \dots, s_{i_l}$.

   So we must have $s_q < s_{i_j}$ or $s_q > s_{i_{j+1}}$, and in either case we include this element so the m-profile does not collapse s_{i_j} and $s_{i_{j+1}}$ into the same element.

2. $s_{i_j}, s_{i_{j+1}}$ are adjacent in $\sigma^{(m)}$, but are not consecutive increasing. Then there exists an s_r in $\sigma^{(m)}$ such that $r < i_j$ or $r > i_{j+1}$ and $s_{i_j} < s_r < s_{i_{j+1}}$, and we insert this s_r into π. Note we only need to add one such s_r.

3. $s_{i_j}, s_{i_{j+1}}$ are adjacent and consecutive increasing in $\sigma^{(m)}$. Previously this case was impossible, but now it can happen and we must add elements to avoid $s_{i_j}, s_{i_{j+1}}$ being collapsed into a single element when we consider the m-profile.

   Since $s_{i_j}, s_{i_{j+1}}$ are adjacent in $\sigma^{(m)}$ and consecutive increasing, by our construction of $\sigma^{(m)}$ we observe $\iota_{i_j} = [1, 2, \ldots, m]$. Thus, as π need only be contained in σ, and not $\sigma^{(m)}$, we can take $m - 1$ consecutive increasing elements which lie between s_{i_j} and $s_{i_{j+1}}$ in σ. This gives us a run of $m + 1$ elements which, in the m-profile, will collapse to two consecutive increasing blocks.

The π thus constructed clearly obeys $\pi \preccurlyeq \sigma$ and $\tau \preccurlyeq \pi^{(m)}$, and in our construction we have at most $m-1$ elements included from each pair t_j, t_{j+1}. Thus

$$|\pi| \leq |\tau| + (m - 1)(|\tau| - 1) = m(|\tau| - 1) + 1$$

□

The finite basis property on $X \wr I_m$ is now simply a corollary of these two lemmas and its proof takes exactly the same steps as Theorem 3.1.3.

Theorem 3.2.4. *If X is a finitely based closed class then $X \wr I_m$ is finitely based for all $m \in \mathbb{N}$.*

Proof. Let $\sigma \in \mathcal{B}(X \wr I_m)$. Then $\sigma^{(m)} \notin X$ and so there exists a $\beta \in \mathcal{B}(X)$ such that $\beta \preccurlyeq \sigma^{(m)}$. We now construct a π minimal such that

$$\beta \preccurlyeq \pi^{(m)} \preccurlyeq \pi \preccurlyeq \sigma$$

which we may do by Lemma 3.2.3. Moreover, $|\pi|$ is bounded in terms of $|\beta|$. Now we have $\pi^{(m)} \succcurlyeq \beta \in \mathcal{B}(X)$, and so by Lemma 3.2.2, $\pi \notin X \wr I_m$. But $\pi \preccurlyeq \sigma \in \mathcal{B}(X \wr I_m)$ and so we must have $\pi = \sigma$ and hence $|\sigma|$ is bounded in terms of $|\beta|$. □

3.3 Finite Basis Property of $X \wr R$ and $X \wr (I \cup R)$

The method we have developed for showing finite basis conditions can be applied to any wreath product $X \wr Y$ where we have an equivalent notion of the *profile* of permutations with respect to elements of Y. Here we exhibit two further examples which use the profiles defined in Section 2.7.1.

The first is $X \wr R$, which uses the skew profile $\sigma^{\downarrow}$ of σ. Since there is a certain symmetry between $X \wr R$ and $X \wr I$, the proof of the finite basis property on $X \wr I$ given in [8] (and replicated here in Section 2.8.2) can be easily adapted to show the following theorem. We omit the proof.

Theorem 3.3.1. *If X is a finitely based closed class then so is $X \wr R$.*

More interesting is the case of the wreath product $X \wr (I \cup R)$. Elements of this wreath product are formed by taking elements of X and expanding each symbol into either a consecutive increasing segment or a consecutive decreasing segment. As we now have finite basis properties for both $X \wr I$ and $X \wr R$ we might hope that $X \wr (I \cup R)$ has a similar property. We can show that this is indeed the case, and to do this we require the strong profile, $\sigma^{\updownarrow}$ of σ, as defined in Section 2.7.1.

As before, we require two technical results to establish the finite basis property. The first is an extension of Lemma 3.1.1 together with its $X \wr R$ equivalent.

Lemma 3.3.2. *$\sigma \in X \wr (I \cup R)$ if and only if $\sigma^{\updownarrow} \in X$.*

Proof. ($\Leftarrow$) Trivial as $\sigma = \sigma^{\updownarrow} \wr (\phi_1, \ldots, \phi_k)$ with each $\phi_i \in I \cup R$.

($\Rightarrow$) We decompose $\sigma = \sigma_1 \ldots \sigma_k$ into maximal consecutive increasing or decreasing segments, and write the strong profile $\sigma^{\updownarrow} = s_1 \ldots s_k$, with $\sigma_i < \sigma_j \Leftrightarrow s_i < s_j$.

Now suppose we have $\tau = t_1 \ldots t_l \in X$ such that $\sigma = \tau \wr (\phi_1, \ldots, \phi_l)$ where each ϕ_i is in R or in I. Then we can partition σ in terms of τ as $\sigma = \tau_1 \ldots \tau_l$ with $\tau_i < \tau_j \Leftrightarrow t_i < t_j$, and each τ_i being a consecutive increasing or a consecutive decreasing segment.

Since our original partition $\sigma_1 \ldots \sigma_k$ had k minimal, we know $l \geq k$. We now claim that we can partition each σ_j into segments from $\tau_1, \ldots, \tau_l$. For if there exists a τ_i that overlaps two segments σ_j and σ_{j+1}, then since τ_i is consecutive increasing or consecutive decreasing we must have $\sigma_j\sigma_{j+1}$ making up a consecutive increasing or a consecutive decreasing segment respectively, contradicting the maximality of $|\sigma_j|$ and $|\sigma_{j+1}|$. Thus we choose $i_1, i_2, \ldots, i_k = l$ such that

$$\begin{aligned} \sigma_1 &= \tau_1 \ldots \tau_{i_1} \\ \sigma_2 &= \tau_{i_1+1} \ldots \tau_{i_2} \\ &\vdots \\ \sigma_k &= \tau_{i_{k-1}+1} \ldots \tau_l \end{aligned}$$

where each σ_j is consecutive increasing or consecutive decreasing.

Now we construct the mapping $s_j \mapsto t_{i_j}$ which is order-preserving:

$$\begin{aligned} s_p < s_q &\iff \sigma_p < \sigma_q \\ &\iff \tau_{i_p} < \tau_{i_q} \text{ since } \tau_{i_p} \preccurlyeq \sigma_p \text{ and } \tau_{i_q} \preccurlyeq \sigma_q \\ &\iff t_{i_p} < t_{i_q} \end{aligned}$$

and so the strong profile $\sigma^{\updownarrow} \preccurlyeq \tau \in X$, and by the closure of X we have $\sigma^{\updownarrow} \in X$. □

The second result is similar to Lemma 3.1.2 but here we have several extra cases to consider to include reverse permutations and possible instances of consecutive elements.

Lemma 3.3.3. *If σ is a permutation such that $\tau \preccurlyeq \sigma^{\updownarrow}$, then if π is minimal subject to*

$$\tau \preccurlyeq \pi^{\updownarrow} \preccurlyeq \pi \preccurlyeq \sigma$$

then $|\pi| \leq 2|\tau| - 1$.

Proof. We proceed as before. Write $\sigma^{\updownarrow} = s_1 \ldots s_l$, $\tau = t_1 \ldots t_k$ and suppose that $s_{i_1}, \ldots, s_{i_k}$ is the lexicographically leftmost occurrence of τ in $\sigma^{\updownarrow}$. We construct a new subsequence π of σ by including the elements $s_{i_1}, \ldots, s_{i_k}$, and for each pair t_j, t_{j+1} we add 0 or 1 symbols from σ to ensure $\pi^{\updownarrow}$ involves the sequence $s_{i_1}, \ldots, s_{i_k}$.

1. t_j, t_{j+1} not consecutive then we need add no further symbols.
2. t_j, t_{j+1} consecutive increasing, then one of the following will occur:
   (a) $s_{i_j}, s_{i_{j+1}}$ are not adjacent in $\sigma^{\updownarrow}$. Then we must have a s_q in $\sigma^{\updownarrow}$ with $i_j < q < i_{j+1}$ and $s_q < s_{i_j}$ or $s_q > s_{i_{j+1}}$, which we include.
   (b) $s_{i_j}, s_{i_{j+1}}$ are adjacent in $\sigma^{\updownarrow}$ but not consecutive increasing. Then there exists s_r in $\sigma^{\updownarrow}$ with $r < i_j$ or $r > i_{j+1}$ and $s_{i_j} < s_r < s_{i_{j+1}}$ which we include in π.
   (c) $s_{i_j}, s_{i_{j+1}}$ adjacent and consecutive increasing in $\sigma^{\updownarrow}$. This tells us that in σ, s_{i_j} was part of a consecutive decreasing segment which was collapsed into a single point when taking the strong profile. We can therefore skew expand the point s_{i_j} into two points $s_{i_j} + 1, s_{i_j}$. The resulting skew expansion is still involved in σ, and $s_{i_j}, s_{i_{j+1}}$ are no longer represented by the same block in $\pi^{\updownarrow}$.
3. t_j, t_{j+1} are consecutive decreasing. We omit the case-by-case study of this as it is analogous to the consecutive increasing case.

We have added at most 1 element to π per pair of elements t_j, t_{j+1}, and from this we obtain the required bound. □

Theorem 3.3.4. *If X is a finitely based closed class then so is $X \wr (I \cup R)$.*

Proof. Let $\sigma \in \mathcal{B}(X \wr (I \cup R))$. Then $\sigma^{\updownarrow} \notin X$ and so there exists a $\beta \in \mathcal{B}(X)$ such that $\beta \preccurlyeq \sigma^{\updownarrow}$. We now construct a π minimal such that

$$\beta \preccurlyeq \pi^{\updownarrow} \preccurlyeq \pi \preccurlyeq \sigma$$

which we may do by Lemma 3.3.3. Moreover, $|\pi|$ is bounded in terms of $|\beta|$. Now we have $\pi^{\updownarrow} \succcurlyeq \beta \in \mathcal{B}(X)$, and so by Lemma 3.3.2, $\pi \notin X \wr (I \cup R)$. But

$\pi \preccurlyeq \sigma \in \mathcal{B}(X \wr (I \cup R))$ and so we must have $\pi = \sigma$ and hence $|\sigma|$ is bounded in terms of $|\beta|$. $\square$

3.4 Finite Basis Property of $X \wr R_n$ and $X \wr (I_m \cup R_n)$

The modification of the proof of the finite basis property for $X \wr I_m$ to cover $X \wr R_n$ is analogous to the modification from $X \wr I$ to $X \wr R$, and as such we will omit the proofs. Again we assume $n \geq 2$ as the case $n = 1$ is trivial: $X \wr R_1 = X$. We first define the concept of the skew n-profile.

Definition 3.4.1. Let σ be any permutation. Then the *skew n-profile*, $\sigma^{\downarrow n} = s_1 s_2 \dots s_k$, of σ is the permutation with k minimal, subject to

$$\sigma = \sigma^{\downarrow n} \wr (\rho_1, \rho_2, \dots, \rho_k)$$

with each $\rho_j \in R_n$.

Where $\sigma^{\downarrow n}$ has two adjacent elements s_i, s_{i+1} such that $s_{i+1} = s_i - 1$ then we take $\rho_i = [n, n-1, \dots, 1]$, the maximal possible permutation in R_n. This ensures our choices of ρ_j are fixed, and thus adds uniqueness to our construction.

It is then an exercise in modifying Lemmas 3.2.2 and 3.2.3 to prove the following two lemmas.

Lemma 3.4.2. *$\sigma \in X \wr R_n$ if and only if $\sigma^{\downarrow n} \in X$.*

Lemma 3.4.3. *Let σ be any permutation and $\tau \preccurlyeq \sigma^{\downarrow n}$. Then if π is minimal subject to*

$$\tau \preccurlyeq \pi^{\downarrow n} \preccurlyeq \pi \preccurlyeq \sigma$$

we have $|\pi| \leq n(|\tau| - 1) + 1$.

The finite basis property then follows as before:

Theorem 3.4.4. *If X is a finitely based closed class then so is $X \wr R_n$.*

We now turn our attention to wreath products with the class $I_m \cup R_n$ for all integers $m, n \geq 2$. If either of m or n equals 1 then the problem reduces to one that we have already covered. First we define a new profile:

Definition 3.4.5. Let σ be any permutation. Then the *strong m, n-profile*, $\sigma^{\updownarrow_n^m} = s_1 s_2 \dots s_k$, of σ is the permutation with k minimal, subject to

$$\sigma = \sigma^{\updownarrow_n^m} \wr (\phi_1, \phi_2, \dots, \phi_k)$$

with each $\phi_j \in I_m \cup R_n$.

Where there exists a consecutive increasing (resp. consecutive decreasing) segment ϕ of σ such that $|\phi| > m$ (resp. $|\phi| > n$) then for suitable i and $j = \lfloor \frac{|\phi|}{m} \rfloor$ (resp. $j = \lfloor \frac{|\phi|}{n} \rfloor$) we write $\phi = \phi_i \phi_{i+1} \dots \phi_{i+j}$ with $|\phi_i|, \dots, |\phi_{i+j-1}| = m$ and $|\phi_{i+j}| \leq m$ (resp. $|\phi_i|, \dots, |\phi_{i+j-1}| = n$ and $|\phi_{i+j}| \leq n$). This ensures our construction of the strong m, n-profile is unique.

Now we extend Lemma 3.2.2 to give our first technical result.

Lemma 3.4.6. *$\sigma \in X \wr (I_m \cup R_n)$ if and only if $\sigma^{\updownarrow_n^m} \in X$.*

Proof. ($\Leftarrow$) Trivial as $\sigma = \sigma^{\updownarrow_n^m} \wr (\phi_1, \dots, \phi_k)$ with each $\phi_j \in I_m \cup R_n$.

($\Rightarrow$) We decompose $\sigma = \sigma_1 \dots \sigma_k$ into maximal consecutive increasing or consecutive decreasing segments as defined by the strong m, n-profile, and write the strong m, n-profile $\sigma^{\updownarrow_n^m} = s_1 \dots s_k$, with $\sigma_i < \sigma_j \Leftrightarrow s_i < s_j$. Moreover where $s_{i+1} = s_i + 1$ and σ_i, σ_{i+1} are both consecutive increasing then $|\sigma_i| = m$, and where $s_{i+1} = s_i - 1$ and σ_i, σ_{i+1} are both consecutive decreasing then $|\sigma_i| = n$.

Now suppose we have $\tau = t_1 \dots t_l \in X$ such that $\sigma = \tau \wr (\phi_1, \dots, \phi_l)$ for some $\phi_1, \dots, \phi_l \in I_m \cup R_n$. Then we can partition σ in terms of τ as $\sigma = \tau_1 \dots \tau_l$ with $\tau_i < \tau_j \Leftrightarrow t_i < t_j$ for all i, j. For all i, whenever τ_i is consecutive increasing then $|\tau_i| \leq m$, and whenever τ_i is consecutive decreasing then $|\tau_i| \leq n$.

Since our original partition $\sigma_1 \dots \sigma_k$ had k minimal, we know $l \geq k$. We now claim that we can partition each σ_j into segments or parts of segments from $\tau_1, \dots, \tau_k$, but in particular that for each σ_j there exists a τ_{i_j} whose right hand end lies within σ_j.

Recall the $\mathcal{L}(\pi)$ and $\mathcal{R}(\pi)$ notation introduced in the proof of Lemma 3.2.2. As in that lemma we write each σ_j in terms of the τ_i and parts thereof:

$$\sigma_j = \mathcal{R}(\tau_{i_{j-1}+1})\tau_{i_{j-1}+2} \dots \tau_{i_j}\mathcal{L}(\tau_{i_j+1})$$

and again we claim that every σ_j contains the right hand end of the permutation τ_{i_j}. If not, then σ_j is properly contained in some τ_p, in which case $|\tau_p| > |\sigma_j| = m$ if σ_j and τ_p are consecutive increasing, or $|\tau_p| > |\sigma_j| = n$ if σ_j and τ_p are consecutive decreasing, both of which are contradictions.

Thus for all $j = 1, \dots, k$, the non-empty segment $\mathcal{R}(\tau_{i_j})$ is contained within σ_j, and so we can construct the order-preserving mapping $s_j \mapsto t_{i_j}$. For all $p, q = 1, \dots, k$:

$$\begin{aligned} s_p < s_q &\iff \sigma_p < \sigma_q \\ &\iff \tau_{i_p} < \tau_{i_q} \text{ since } \mathcal{R}(\tau_{i_p}) \text{ lies in } \sigma_p \text{ and } \mathcal{R}(\tau_{i_q})\text{lies in } \sigma_q \\ &\iff t_{i_p} < t_{i_q} \end{aligned}$$

and so the strong m, n-profile $\sigma^{\updownarrow^m_n} \preccurlyeq \tau \in X$, and by the closure of X we have $\sigma^{\updownarrow^m_n} \in X$. $\square$

The second result is similar to Lemma 3.2.3, although here we must consider a number of different cases.

Lemma 3.4.7. *Let σ be any permutation and $\tau \preccurlyeq \sigma^{\updownarrow^m_n}$. Then if π is minimal subject to*

$$\tau \preccurlyeq \pi^{\updownarrow^m_n} \preccurlyeq \pi \preccurlyeq \sigma$$

we have $|\pi| \leq \max(m, n)(|\tau| - 1) + 1$.

Proof. Let $\tau = t_1 \dots t_l$, $\sigma^{\updownarrow^m_n} = s_1 \dots s_k$ and write $\sigma = \sigma^{\updownarrow^m_n} \wr (\phi_1, \dots, \phi_k)$ with each $\phi_j \in I_m \cup R_n$ subject to satisfying our conditions in Definition 3.4.5. Thus we have the partition $\sigma = \sigma_1 \dots \sigma_k$ with $\sigma_i < \sigma_j \Leftrightarrow s_i < s_j$.

Let us also suppose that $s_{i_1}, \dots, s_{i_l}$ is the lexicographically leftmost occurrence of τ in $\sigma^{\updownarrow^m_n}$. We construct our new permutation π by first including all of $s_{i_1}, \dots, s_{i_l}$, and for each pair t_j, t_{j+1} we add a further 0, 1, $m-1$ or $n-1$ symbols as follows:

1. t_j, t_{j+1} are not consecutive increasing or consecutive decreasing then we do not need to add anything.

2. t_j, t_{j+1} are consecutive increasing. There are 3 possible cases:

   (a) $s_{i_j}, s_{i_{j+1}}$ are not adjacent in $\sigma^{\updownarrow^m_n}$, so there exists a symbol s_q such that $i_j < q < i_{j+1}$. If this symbol is such that $s_{i_j} < s_q < s_{i_{j+1}}$ then we could have matched t_{j+1} to s_q rather than $s_{i_{j+1}}$, contradicting our leftmost choice of $s_{i_1}, \dots, s_{i_l}$.

   So we must have $s_q < s_{i_j}$ or $s_q > s_{i_{j+1}}$, and in either case we include this element so that the m-profile does not collapse s_{i_j} and $s_{i_{j+1}}$ into the same element.

   (b) $s_{i_j}, s_{i_{j+1}}$ are adjacent in $\sigma^{\updownarrow^m_n}$, but are not consecutive increasing. Then there exists an s_r in $\sigma^{\updownarrow^m_n}$ such that $r < i_j$ or $r > i_{j+1}$ and $s_{i_j} < s_r < s_{i_{j+1}}$, and we insert this s_r into π. Note we only need to add one such s_r.

   (c) $s_{i_j}, s_{i_{j+1}}$ are adjacent and consecutive increasing in $\sigma^{\updownarrow^m_n}$. s_{i_j} and $s_{i_{j+1}}$ come from the segments σ_{i_j} and $\sigma_{i_{j+1}}$ of σ respectively, and we now need to add some (or all) of σ_{i_j} and $\sigma_{i_{j+1}}$ to π, observing that this still ensures $\pi \preccurlyeq \sigma$. These segments can be consecutive increasing or consecutive decreasing, which gives rise to 4 different cases:

- If both σ_{i_j} and $\sigma_{i_{j+1}}$ are consecutive increasing, then by our construction of $\sigma^{\updownarrow^m_n}$ we have $|\sigma_{i_j}| = m$ and so expanding s_{i_j} into σ_{i_j} gives us, together with $s_{i_{j+1}}$, a run of $m+1$ consecutive increasing elements which ensures that s_{i_j} and $s_{i_{j+1}}$ appear in different symbols in the strong m,n-profile $\pi^{\updownarrow^m_n}$. As $\sigma_{i_j} \preccurlyeq \sigma$ we still have $\pi \preccurlyeq \sigma$.
- If σ_{i_j} is consecutive increasing but $\sigma_{i_{j+1}}$ is consecutive decreasing then we skew expand the symbol $s_{i_{j+1}}$ into two points in π (taken from the decreasing sequence $\sigma_{i_{j+1}}$), regarding the extra point to lie between s_{i_j} and $s_{i_{j+1}}$.
- The final two cases can be dealt with together. They are where σ_{i_j} is consecutive decreasing, and $\sigma_{i_{j+1}}$ is consecutive increasing or decreasing. In both cases we simply expand s_{i_j} into two consecutive decreasing points from σ_{i_j} in π.

3. If t_j, t_{j+1} are consecutive decreasing then the construction is similar to case 2.

The π constructed clearly obeys $\pi \preccurlyeq \sigma$ and $\tau \preccurlyeq \pi^{\updownarrow^m_n}$, and in our construction we have added at most $\max(m-1, n-1)$ elements to each pair t_j, t_{j+1}. Thus

$$|\pi| \leq |\tau| + \max(m-1, n-1)(|\tau|-1) = \max(m,n)(|\tau|-1)+1$$

□

As before, we can now derive the finite basis property for $X \wr (I_m \cup R_n)$.

Theorem 3.4.8. *If X is a finitely based closed class then $X \wr (I_m \cup R_n)$ is finitely based for all $m, n \in \mathbb{N}$.*

Proof. Let $\sigma \in \mathcal{B}(X \wr (I_m \cup R_n))$. Then $\sigma^{\updownarrow^m_n} \notin X$ and so there exists $\beta \in \mathcal{B}(X)$ such that $\beta \preccurlyeq \sigma^{\updownarrow^m_n}$. We now construct a π minimal such that

$$\beta \preccurlyeq \pi^{\updownarrow^m_n} \preccurlyeq \pi \preccurlyeq \sigma$$

which we may do by Lemma 3.4.7. Moreover, $|\pi|$ is bounded in terms of $|\beta|$. Now we have $\pi^{\updownarrow_n^m} \succcurlyeq \beta \in \mathcal{B}(X)$, and so by Lemma 3.4.6, $\pi \notin X \wr (I_m \cup R_n)$. But $\pi \preccurlyeq \sigma \in \mathcal{B}(X \wr (I_m \cup R_n))$ and so we must have $\pi = \sigma$ and hence $|\sigma|$ is bounded in terms of $|\beta|$. □

3.5 Remarks and Conjectures

We complete this thesis by considering other classes Y whose wreath product $X \wr Y$ has the finite basis property, and later where Y is constructed from classes we have already considered. It is clear that if X is finitely based we only need to develop some equivalent notion of a unique profile for Y to attempt to establish the finite basis property. In general the method is likely to fail if we cannot obtain a bound on how large the basis elements need to be.

Finite Classes If we take $Y = F$, some arbitrary finite class, we can immediately get a bound on the size of basis elements of $X \wr F$ providing we can define an *F-profile* based on permutations in F. Our choice of F-profile needs to be such that we have the two way implication

$$\sigma \in X \wr F \Longleftrightarrow \sigma^{(F)} \in X.$$

At present it is unknown whether this is obtainable. The author believes that a suitable choice of F-profile should yield this result, from which we would obtain:

Conjecture 3.5.1. *Let F be any finite class X an arbitrary closed class. If X is finitely based then $X \wr F$ is also finitely based.*

Unions There seems to be no problem with taking the union of any classes so far considered, as we saw with $Y = I \cup R$ and $Y = I_m \cup R_n$. In this vein $I \cup R_n$ and $I_m \cup R$ look like they should also be possible. In this case, we have that for all classes Y and Z whose wreath products $X \wr Y$ and $X \wr Z$ are already known to have the finite basis property then their union also has the finite basis property, which gives evidence to support the following:

Conjecture 3.5.2. *If Y and Z are closed classes such that $X \wr Y$ and $X \wr Z$ are finitely based for all finitely based classes X, then $X \wr (Y \cup Z)$ is also finitely based.*

Note that if Conjecture 3.5.1 holds, then for any two finite classes F and G, their union $F \cup G$ is also finite and hence $X \wr (F \cup G)$ would have the finite basis property, which would further support this claim.

Ticks and Direct Sums Another case where we may have the finite basis property is for classes which have only a finite number of shapes that permutations can take. For example, it seems likely that $Y = \mathrm{Sub}(21345\ldots) = \{1, 12, 21, 123, 213, \ldots\}$ will give the finite basis property. Indeed we can also probably extend this to classes with a tick of size n at the beginning, $Y = \mathrm{Sub}([n, n-1, \ldots, 2, 1, n+1, n+2, \ldots])$:

Conjecture 3.5.3. *Let X be any finitely based closed class. Then $X \wr \mathrm{Sub}([n, n-1, \ldots, 1, n+1, n+1, \ldots])$ is also finitely based.*

It is interesting to note here that classes with an n-tick at the beginning before an increasing permutation can be written as a direct sum $Y = R_n \oplus I$ and we may wonder whether direct sums of other classes already considered give us the finite basis property. Cases to investigate could be $I_n \oplus R$ and $I \oplus R$. We may also consider skew sums, some of which are likely to follow by symmetry arguments.

Conjecture 3.5.4. *Let X be any finitely based closed class. Then $X \wr (I \oplus R)$ and $X \wr (I_n \oplus R)$ are also finitely based.*

Other Constructions The other constructions introduced in Chapter 2 seem less likely to succeed, or are relatively trivial. Intersection reduces to a trivial case in most of the classes we have considered so far, and differentiation is also straightforward (note, for example, $\partial I_m = I_{m-1}$).

Juxtaposition and merges seem less likely to succeed although possibly $[I, I]$ might be a starting point to consider.

The wreath product itself is more interesting. For example taking $Y = I \wr R$ $Y = I \wr R_n$ or $Y = I_m \wr R$ may result in the finite basis property in some cases. Such results are unlikely to be straightforward.

Conjecture 3.5.5. *Let X be any finitely closed class. Then $X \wr (I \wr R)$, $X \wr (I \wr R_n)$ and $X \wr (I_m \wr R)$ are also finitely based.*

Bibliography

[1] M.H. Albert and M.D. Atkinson, *Simple permutations, partial well-order, and enumeration*, Proceedings, Permutation Patterns 2003 (M.H. Albert, ed.), http://www.cs.otago.ac.nz/trseries/oucs-2003-02.pdf, 2003, pp. 5–9.

[2] M.H. Albert, M.D. Atkinson, and M. Klazar, *The enumeration of simple permutations*, Journal of Integer Sequences **6** (2003), 03.4.4.

[3] M.H. Albert, M.D. Atkinson, and N. Ruškuc, *Juxtaposition of pattern avoidance classes*, In preparation (2004).

[4] M.D. Atkinson, *Restricted permutations*, Discrete Mathematics **195** (1999), 27–38.

[5] M.D. Atkinson and R. Beals, *Permuting Mechanisms and Closed Classes of Permutations*, Combinatorics, Computation and Logic (1999), 117–127.

[6] ———, *Permutation Involvement and Groups*, Quart. J. Math. **52** (2001), 415–421.

[7] M.D. Atkinson, M.M. Murphy, and N. Ruškuc, *Pattern avoidance classes and subpermutations*, Submitted (2004).

[8] M.D. Atkinson and T. Stitt, *Restricted permutations and the wreath product*, Discrete Mathematics **259** (2002), 19–36.

[9] M Bóna, *A walk through combinatorics: an introduction to enumeration and graph theory*, River Edge, NJ: World Scientific, 2002.

[10] D.E. Knuth, *Fundamental Algorithms, the Art of Computer Programming*, 2 ed., vol. 1, Addison-Wesley, Reading, MA, 1973.

[11] A. Marcus and G. Tardos, *Excluded permutation matrices and the Stanley-Wilf conjecture*, J. Comb. Theory Ser. A **107** (2004), 153–160.

[12] M.M. Murphy, *Restricted Permutations, Antichains, Atomic Classes and Stack Sorting*, Ph.D. thesis, University of St Andrews, 2002.