SIMPLE PERMUTATIONS AND ALGEBRAIC GENERATING FUNCTIONS

ROBERT BRIGNALL, SOPHIE HUCZYNSKA*, AND VINCENT VATTER†
School of Mathematics and Statistics
University of St Andrews
St Andrews, Fife, Scotland
{robertb, sophieh, vince}@mcs.st-and.ac.uk
http://turnbull.mcs.st-and.ac.uk/~{robertb, sophieh, vince}

A simple permutation is one that never maps a nontrivial contiguous set of indices contiguously. Given a set of permutations that is closed under taking subpermutations and contains only finitely many simple permutations, we provide a framework for enumerating subsets that are restricted by properties belonging to a finite "query-complete set". Such properties include being even, being an alternating permutation, and avoiding a given generalised (blocked or barred) pattern. We show that the generating functions for these subsets are always algebraic, thereby generalising recent results of Albert and Atkinson. We also apply these techniques to the enumeration of involutions and cyclic closures.

1. INTRODUCTION

Substitution decompositions (known also as modular decompositions, disjunctive decompositions, and X-joins) have proved to be a useful technique in a wide range of settings, ranging from game theory to combinatorial optimization, see Möhring [26] or Möhring and Radermacher [27] for extensive references. Although substitution decompositions are most often applied to algorithmic problems, here we apply them enumeratively.

An *interval* in the permutation π is a set of contiguous indices $I = [a, b]$ such that the set of values $\pi(I) = \{\pi(i) : i \in I\}$ is also contiguous. Every permutation π of $[n] = \{1, 2, \ldots, n\}$ has intervals of length 0, 1, and n; π is said to be *simple* if it has no other intervals (such intervals are called *proper*). Figure 1 shows three simple permutations.

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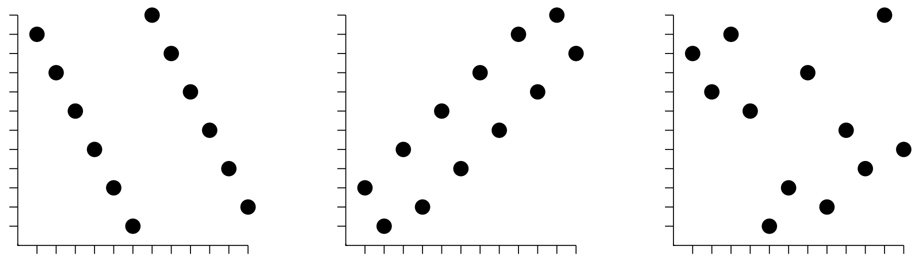

Figure 1: The plots of three simple permutations of length 12.

Our aim is to enumerate subsets of certain sets of permutations closed under taking subpermutations; the permutation π is said to *contain* the permutation σ, written $\sigma \leq \pi$, if π has a subsequence that is order isomorphic to σ, and otherwise π is said to *avoid* σ. For example, $\pi = 491867532$ contains $\sigma = 51342$, as can be seen by considering the subsequence 91672 $(= \pi(2), \pi(3), \pi(5), \pi(6), \pi(9))$. This pattern-containment relation is a partial order on permutations, and we refer to downsets of permutations under this order as *permutation classes*. In other words, if $\mathcal{C}$ is a permutation class, $\pi \in \mathcal{C}$, and $\sigma \leq \pi$, then $\sigma \in \mathcal{C}$.

We denote by $\mathcal{C}_n$ the set $\mathcal{C} \cap S_n$, i.e. the permutations in $\mathcal{C}$ of length n, and we refer to $\sum |\mathcal{C}_n| x^n$ as the *generating function for* $\mathcal{C}$. For any permutation class $\mathcal{C}$, there is a unique antichain[1] B such that $\mathcal{C}$ consists of every permutation that contains no element of B, i.e., $\mathcal{C} = \{\pi : \beta \not\leq \pi \text{ for all } \beta \in B\}$, which we abbreviate to $\mathrm{Av}(B)$. The antichain B, which comprises the minimal permutations not in $\mathcal{C}$, is called the *basis* of $\mathcal{C}$.

Our main theorem appears below; the definition of query-complete sets of properties follows.

Theorem 1.1. *Let $\mathcal{C}$ be a permutation class containing only finitely many simple permutations, $\mathcal{P}$ a finite query-complete set of properties, and $\mathcal{Q} \subseteq \mathcal{P}$. The generating function for the set of permutations in $\mathcal{C}$ satisfying every property in $\mathcal{Q}$ is algebraic over $\mathbb{Q}[x]$.*

One class to which this theorem applies is $\mathrm{Av}(132)$. In any permutation from $\mathrm{Av}(132)$, all entries to the left of the maximum must be greater than all entries to the right. This shows that $\mathrm{Av}(132)$ has only three simple permutations (1, 12, and 21).

Given $\sigma \in S_m$ and nonempty permutations $\alpha_1, \ldots, \alpha_m$, the *inflation* of σ by $\alpha_1, \ldots, \alpha_m$ — denoted $\sigma[\alpha_1, \ldots, \alpha_m]$ — is the permutation obtained by replacing each entry $\sigma(i)$ by an interval that is order isomorphic to α_i. For example, $2413[1, 132, 321, 12] = 479832156$ (see Figure 2). Simple permutations cannot be deflated. Conversely:

Proposition 1.2 (Albert and Atkinson [2]). *Every permutation except 1 is the inflation of a unique simple permutation of length at least 2.*

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permutation, simple permutation, substitution decomposition

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[1]Recall that an *antichain* is a set of pairwise incomparable elements.

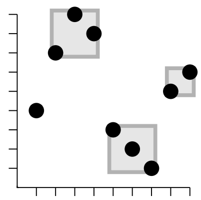

Figure 2: The plot of 479832156, an inflation of 2413.

Sketch of proof. Consider the intervals of the permutation π which are not contained in any other proper intervals. If these intervals are disjoint, then the Proposition clearly holds. Otherwise two of these intervals, say I and J, intersect; however, $I \cup J$ must then also be an interval, and by maximality, $I \cup J$ must therefore contain every entry of π. It is easy at this point to see that π is the inflation of either 12 or 21. ☐

A *property*, P, is any set of permutations[2]. We say that π *satisfies* P if $\pi \in P$. We define a set $\mathcal{P}$ of properties to be *query-complete* if, for each simple permutation σ of length m and property $P \in \mathcal{P}$, there is a procedure to determine whether $\sigma[\alpha_1, \ldots, \alpha_m]$ satisfies P which requires only knowledge of which properties of $\mathcal{P}$ each α_i satisfies. For example, the set of properties consisting of the 132-avoiding permutations, $\{\mathrm{Av}(132)\}$, is not query-complete, as witnessed by the fact that $12[1,1] \in \mathrm{Av}(132)$ but $12[1,21] \notin \mathrm{Av}(132)$, while both 1 and 12 avoid 132. However, $\{\mathrm{Av}(132), \mathrm{Av}(21)\}$ is query-complete:

$$\begin{aligned}
12[\alpha_1, \alpha_2] \in \mathrm{Av}(132) \quad &\Longleftrightarrow \quad \alpha_1 \in \mathrm{Av}(132) \text{ and } \alpha_2 \in \mathrm{Av}(21),\\
21[\alpha_1, \alpha_2] \in \mathrm{Av}(132) \quad &\Longleftrightarrow \quad \alpha_1 \in \mathrm{Av}(132) \text{ and } \alpha_2 \in \mathrm{Av}(132),\\
\sigma[\alpha_1, \ldots, \alpha_m] \notin \mathrm{Av}(132) \quad &\text{if} \quad \sigma \notin \{1, 12, 21\} \text{ is simple},\\
12[\alpha_1, \alpha_2] \in \mathrm{Av}(21) \quad &\Longleftrightarrow \quad \alpha_1 \in \mathrm{Av}(21) \text{ and } \alpha_2 \in \mathrm{Av}(21),\\
\sigma[\alpha_1, \ldots, \alpha_m] \notin \mathrm{Av}(21) \quad &\text{if} \quad \sigma \notin \{1, 12\} \text{ is simple}.
\end{aligned}$$

Note that since $\sigma[\alpha_1, \ldots, \alpha_m]$ is uniquely determined by σ and the α_i's, every property P lies in some query-complete set, e.g., $\{P\} \cup \{\{\pi\} : \pi \text{ a permutation}\}$ is query-complete for every P. Thus the finiteness condition in Theorem 1.1 is essential. Another observation about query-complete sets, which will be liberally applied, is the following.

Proposition 1.3. *A union of query-complete sets of properties is itself query-complete.*

The next section establishes various query-complete sets of properties. Section 3 contains the proof of Theorem 1.1 while Section 4 gives numerous examples. In Sections 5 and 6 we adapt these techniques to enumerate involutions and cyclic closures, respectively. We end by discussing the scope of this technique in Section 7.

To demonstrate the applicability of our results, we conclude the introduction by stating the following corollary (the terms contained in it are reviewed as needed in Section 2).

[2]For example, permutation classes are properties. Indeed, the graph theoretic analogues of permutation classes are commonly referred to as hereditary properties.

Corollary 1.4. *In a permutation class $\mathcal{C}$ with only finitely many simple permutations, the generating functions for the following sequences are algebraic over $\mathbb{Q}[x]$:*

- *the number of permutations in $\mathcal{C}_n$ (this is the result of Albert and Atkinson [2]),*
- *the number of alternating permutations in $\mathcal{C}_n$,*
- *the number of even permutations in $\mathcal{C}_n$,*
- *the number of Dumont permutations of the first kind in $\mathcal{C}_n$,*
- *the number of permutations in $\mathcal{C}_n$ avoiding any finite set of blocked or barred permutations, and*
- *the number of involutions in $\mathcal{C}_n$.*

Moreover, these conditions can be combined in any finite manner desired.

As mentioned previously, $\mathrm{Av}(132)$ contains only three simple permutations, so Corollary 1.4 explains, e.g., why the even permutations in $\mathrm{Av}(132, \beta)$ have an algebraic generating function for every β, first proved in Mansour [24]. Other results in the literature to which Corollary 1.4 applies appear in [11, 12, 13, 15, 17, 18, 21, 22, 23, 25].

2. FINITE QUERY-COMPLETE SETS

We exhibit several query-complete sets of properties in this section. The first of these is necessary for the proof of Theorem 1.1, the others for Corollary 1.4.

Lemma 2.1. *For every permutation β, the set $\{\mathrm{Av}(\delta) : \delta \leq \beta\}$ is query-complete.*

Proof. We prove the lemma by induction on the length of β. The base case $\beta = 1$ being trivial, let us suppose that β is of length at least 2. By induction, $\{\mathrm{Av}(\gamma) : \gamma \leq \delta\}$ is query-complete for all $\delta < \beta$, and thus by appealing to Proposition 1.3 it suffices to prove that whether $\pi = \sigma[\alpha_1, \ldots, \alpha_m]$ satisfies $\mathrm{Av}(\beta)$ can be decided entirely by knowing, for each i, which permutations δ satisfy $\delta \leq \alpha_i$ and $\delta \leq \beta$.

We define a *lenient inflation* to be an inflation $\sigma[\gamma_1, \ldots, \gamma_m]$ in which the γ_i's are allowed to be empty. List all expressions of β as a lenient inflation of σ as

$$\begin{aligned} \beta &= \sigma[\gamma_1^{(1)}, \ldots, \gamma_m^{(1)}], \\ &\vdots \\ \beta &= \sigma[\gamma_1^{(t)}, \ldots, \gamma_m^{(t)}]. \end{aligned}$$

Clearly if we have, for some $s \in [t]$, $\alpha_i \geq \gamma_i^{(s)}$ for all $i \in [m]$, then $\pi \geq \beta$. Equivalently, to have $\pi \in \mathrm{Av}(\beta)$, for every $s \in [t]$ there must be at least one $i \in [m]$ for which $\alpha_i \not\geq \gamma_i^{(s)}$. Conversely, every embedding of β into π gives one of the lenient inflations in the list above, which completing the proof. □

In a *barred permutation*, one or more of the entries is barred; for π to avoid the barred permutation σ means that every set of entries of π order isomorphic to the nonbarred entries of σ can be extended to a set order isomorphic to σ itself. For example, 24315 avoids $21\overline{3}$ because every inversion (i.e., copy of 21) can be extended to a copy of 213 (append the 5), but 24315 contains $3\overline{1}2$ because the 3 and 1 are order isomorphic to 32, but there is no way to extend this to a copy of 312. Barred permutations have arisen several times in the permutation pattern literature. For example, under West's notion of 2-stack sorting [34] the permutations that can be sorted are those that avoid 2341 and $3\overline{5}241$, while Bousquet-Mélou and Butler [6] characterise the permutations corresponding to locally factorial Schubert varieties in terms of barred permutations.

A *blocked permutation* is a permutation containing dashes indicating the entries that need not occur consecutively (in the normal pattern-containment order, no entries need occur consecutively), or in the case of the beginning or trailing dashes, entries that need not occur at the beginning or end of the permutation, respectively. For example, 24135 contains only one copy of -1-23-, namely 235; the entries 245 do not form a copy of -1-23- because the 4 and 5 are not adjacent. Babson and Steingrímsson [5] introduced blocked permutations (although they called them generalised patterns, and implicitly assumed that their patterns had beginning and trailing dashes) and showed that they could be used to express most Mahonian statistics. For example, the major index[3] of π is equal to the total number of copies of -1-32-, -2-31-, -3-21-, and -21- in π.

The proof of Lemma 2.1 extends in a straightforward manner to show that the property of avoiding a blocked or barred permutation (or, for that matter, a permutation combining these restrictions) also lies in a finite query-complete set, although the sets are not so easily described[4].

The permutation $\pi \in S_n$ is said to be *alternating* if for all $i \in [2, n-1]$, $\pi(i)$ does not lie between $\pi(i-1)$ and $\pi(i+1)$.

Lemma 2.2. *The set of properties consisting of*

- $AL = \{$*alternating permutations*$\}$,
- $BR = \{$*permutations beginning with a rise, i.e., permutations with* $\pi(1) < \pi(2)\}$,
- $ER = \{$*permutations ending with a rise*$\}$, *and*
- $\{1\}$.

is query-complete.

[3]The major index is more commonly defined as the sum of the descents of π, $\sum_{\pi(i)>\pi(i+1)} i$.

[4]Consider, e.g., the problem of deciding whether $\pi = 3142[\alpha_1, \alpha_2, \alpha_3, \alpha_4]$ avoids -1-23-. First, each of the α_i's must avoid -1-23-. Then we also need α_3 and α_4 to not contain ascents (i.e., avoid -12-) since α_2 is nonempty, and α_2 to avoid -1-2, since otherwise the third element of the -1-23- could be chosen from α_3.

Proof. Clearly $\{\{1\}, BR, ER\}$ is query-complete:

$$\sigma[\alpha_1,\ldots,\alpha_m] \in BR \iff \alpha_1 \in BR \text{ or } (\alpha_1 = 1 \text{ and } \sigma \in BR)\,,$$
$$\sigma[\alpha_1,\ldots,\alpha_m] \in ER \iff \alpha_m \in ER \text{ or } (\alpha_m = 1 \text{ and } \sigma \in ER)\,.$$

For $\pi = \sigma[\alpha_1,\ldots,\alpha_m]$ to be an alternating permutation, we first need $\alpha_1,\ldots,\alpha_m \in AL$. Now suppose that the entries of π up to and including the $\sigma(i)$ interval are alternating (we have this for $i = 1$ from the above). If $\sigma(i) > \sigma(i+1)$ then π contains a descent between its $\sigma(i)$ interval and its $\sigma(i+1)$ interval. Thus $\sigma(i)$ is allowed to be 1 (i.e., $\sigma \in \{1\}$) only if $i = 1$ or $\sigma(i-1) < \sigma(i)$, while if $\alpha_i \neq 1$ then we must have $\alpha_i \in ER$, and whether or not α_i is 1 we must have $\alpha_{i+1} \in BR \cup \{1\}$. The case where $\sigma(i) < \sigma(i+1)$ is analogous, completing the proof. □

Recall that an *even permutation* is one that can be written as the product of an even number of transpositions, or (much more conveniently for our purposes) a permutation with an even number of inversions.

Lemma 2.3. *The set of properties consisting of*

- $EV = \{$*even permutations*$\}$ *and*
- $EL = \{$*permutations of even length*$\}$

is query-complete.

Proof. We have

$$\sigma[\alpha_1,\ldots,\alpha_m] \in EL \iff \text{an even number of } \alpha_i\text{'s fail to lie in } EL,$$

so $\{EL\}$ is query-complete. To see that $\{EV, EL\}$ is query-complete, we divide the inversions in $\sigma[\alpha_1,\ldots,\alpha_m]$ into two groups: inversions within a single $\sigma(i)$ interval and inversions between two intervals $\sigma(i)$ and $\sigma(j)$. We need to compute the parity of each of these numbers. The parity of the first type of inversions depends only on whether $\alpha_i \in EV$. For the second type, suppose $i < j$. If $\sigma(i) < \sigma(j)$ then there are an even number of inversions (more specifically, 0) between the intervals $\sigma(i)$ and $\sigma(j)$ while if $\sigma(i) > \sigma(j)$ then the number of inversions between these intervals is even if α_i or α_j lie in EL and odd otherwise. □

A permutation is *Dumont of the first kind* if each even entry is immediately followed by a smaller entry and each odd entry is either immediately followed by a larger entry or occurs last (this dates back to Dumont [9]).

Lemma 2.4. *The set of properties consisting of*

- $DU = \{$*Dumont permutations of the first kind*$\}$ *and*
- $EL = \{$*permutations of even length*$\}$

is query-complete.

Proof. It suffices to determine which entries of $\sigma[\alpha_1, \ldots, \alpha_m]$ have even value and which have odd value, and this can be decided based on the knowledge of which α_i's have even length. □

The imaginative reader should at this point have no trouble constructing many other properties that lie in finite query-complete sets. Examples include the property of beginning with a 1, or more generally of mapping any fixed i to any fixed j, or of having major index congruent to $1 \bmod 3$, or having an odd number of left-to-right minimas, or having the repeated pattern of two ascents followed by a descent.

3. PROOF OF MAIN RESULT

We begin by refining Proposition 1.2, which shows that every permutation is the inflation of a unique simple permutation. These propositions follow almost immediately from the proof of Proposition 1.2. Note that there are no simple permutations of length 3, and that 12 and 21 are simple.

Proposition 3.1 (Albert and Atkinson [2])**.** *If π can be written as $\sigma[\alpha_1, \ldots, \alpha_m]$ where σ is simple and $m \geq 4$, then the α_i's are unique.*

In the case where $\pi = 12[\alpha_1, \alpha_2]$, some caution is needed. A *sum indecomposable* permutation is one that cannot be written as $12[\alpha_1, \alpha_2]$ (these are also called connected permutations), whilst a *skew indecomposable* permutation is one that cannot be written as $21[\alpha_1, \alpha_2]$.

Proposition 3.2 (Albert and Atkinson [2])**.** *If π is an inflation of 12, then there is a unique sum indecomposable α_1 such that $\pi = 12[\alpha_1, \alpha_2]$ for some α_2, which is itself unique. The same holds with 12 replaced by 21 and "sum" replaced by "skew".*

We refer to the unique decompositions guaranteed by Propositions 1.2, 3.1, and 3.2 as the *substitution decomposition*.

A class $\mathcal{C}$ of permutations is *wreath-closed*[5] if $\sigma[\alpha_1, \ldots, \alpha_m] \in \mathcal{C}$ for all $\sigma, \alpha_1, \ldots, \alpha_m \in \mathcal{C}$. The *wreath-closure* of a set X, $\mathcal{W}(X)$, is defined as the smallest wreath-closed class containing X. (This concept is well-defined and exists because the intersection of wreath-closed

[5] It is quite easy to decide if a permutation class given by a finite basis is wreath-closed:

Proposition 3.3 (Atkinson and Stitt [3])**.** *A permutation class is wreath-closed if and only if each of its basis elements is simple.*

One may also wish to compute the basis of $\mathcal{W}(\mathcal{C})$. This is routine for classes with finitely many simple permutations (see Proposition 7.3), but much less so in general. In his thesis [28] Murphy gives an example of a finitely based class whose wreath closure is infinitely based. The natural question is then:

Question 3.4. *Given a finite basis B, is it decidable whether $\mathcal{W}(\mathrm{Av}(B))$ is finitely based?*

(See Proposition 7.3 for a special case.)

The analogous question for graphs was raised by Giakoumakis [16] and has received a sizable amount of attention, see for example Zverovich [35].

classes is wreath-closed and the set of all permutations is wreath-closed.) Letting $\mathrm{Si}(X)$ denote the simple permutations in the class $\mathcal{C}$ we see that $\mathrm{Si}(\mathcal{C}) = \mathrm{Si}(\mathcal{W}(\mathcal{C}))$, and indeed $\mathcal{W}(\mathcal{C})$ is the largest class with this property. For example, the wreath closure of $\mathrm{Av}(132)$ is the largest class whose only simple permutations are 1, 12, and 21. This class is known as the *separable permutations*[6], $\mathrm{Av}(2413, 3142)$.

Given a permutation class $\mathcal{C}$ and set $\mathcal{P}$ of properties, we write $\mathcal{C}_{\mathcal{P}}$ for the set of permutations in $\mathcal{C}$ that satisfy every property in $\mathcal{P}$, and write $f_{\mathcal{P}}$ for the generating function of $\mathcal{C}_{\mathcal{P}}$. Before beginning the proof of Theorem 1.1 we consider the case where $\mathcal{C}$ is wreath-closed and $\mathcal{P} = \emptyset$, which contains many of the main ideas of the proof in a more digestible form. (This presentation borrows heavily from Albert and Atkinson [2].)

We begin by introducing two properties,

$$\begin{aligned} \not\oplus &= \{\text{sum indecomposable permutations}\} \text{ and} \\ \not\ominus &= \{\text{skew indecomposable permutations}\}. \end{aligned}$$

Note that both $\{\not\oplus\}$ and $\{\not\ominus\}$ are query-complete, because for simple σ,

$$\begin{aligned} \sigma[\alpha_1, \ldots, \alpha_m] \in \not\oplus &\iff \sigma \neq 12 \text{ and} \\ \sigma[\alpha_1, \ldots, \alpha_m] \in \not\ominus &\iff \sigma \neq 21. \end{aligned}$$

We also introduce the notation

$$\sigma[\mathcal{C}^1, \ldots, \mathcal{C}^m] = \{\sigma[\alpha_1, \ldots, \alpha_m] : \alpha_i \in \mathcal{C}^i \text{ for all } i \in [m]\}.$$

By Propositions 1.2, 3.1, and 3.2 and the assumption that $\mathcal{C}$ is wreath-closed, $\mathcal{C}$ can be written as

$$\mathcal{C} = \{1\} \uplus 12[\mathcal{C}_{\not\oplus}, \mathcal{C}] \uplus 21[\mathcal{C}_{\not\ominus}, \mathcal{C}] \uplus \biguplus_{\substack{\sigma \in \mathrm{Si}(\mathcal{C}) \\ |\sigma| \geq 4}} \sigma[\mathcal{C}, \ldots, \mathcal{C}],$$

while $\mathcal{C}_{\not\oplus}$ and $\mathcal{C}_{\not\ominus}$ have the expressions

$$\begin{aligned} \mathcal{C}_{\not\oplus} &= \{1\} \uplus 21[\mathcal{C}_{\not\ominus}, \mathcal{C}] \uplus \biguplus_{\substack{\sigma \in \mathrm{Si}(\mathcal{C}) \\ |\sigma| \geq 4}} \sigma[\mathcal{C}, \ldots, \mathcal{C}] &= \mathcal{C} \setminus 12[\mathcal{C}_{\not\oplus}, \mathcal{C}], \\ \mathcal{C}_{\not\ominus} &= \{1\} \uplus 12[\mathcal{C}_{\not\oplus}, \mathcal{C}] \uplus \biguplus_{\substack{\sigma \in \mathrm{Si}(\mathcal{C}) \\ |\sigma| \geq 4}} \sigma[\mathcal{C}, \ldots, \mathcal{C}] &= \mathcal{C} \setminus 21[\mathcal{C}_{\not\ominus}, \mathcal{C}]. \end{aligned}$$

[6]The separable permutations seem to have made their first appearance as the permutations that can be sorted by pop-stacks in series, see Avis and Newborn [4]. Shapiro and Stephens [31] showed that the separable permutations are those that fill up under bootstrap percolation. The separable permutations are essentially the permutation analogue of series-parallel posets (see Stanley [32, Section 3.2]) and complement reducible graphs (see Corneil, Lerchs, and Burlingham [8]). Their enumeration is given by the large Schröder numbers (see Footnote 7 or Example 4.1).

These give the system

$$\begin{cases} f &=& x + f_{\not\oplus} f + f_{\not\ominus} f + \sum\limits_{\substack{\sigma \in \mathrm{Si}(\mathcal{C}) \\ |\sigma| \geq 4}} f^{|\sigma|}, \\ f_{\not\oplus} &=& x + f_{\not\ominus} f + \sum\limits_{\substack{\sigma \in \mathrm{Si}(\mathcal{C}) \\ |\sigma| \geq 4}} f^{|\sigma|} &=& f - f_{\not\oplus} f &=& \dfrac{f}{1+f}, \\ f_{\not\ominus} &=& x + f_{\not\oplus} f + \sum\limits_{\substack{\sigma \in \mathrm{Si}(\mathcal{C}) \\ |\sigma| \geq 4}} f^{|\sigma|} &=& f - f_{\not\ominus} f &=& \dfrac{f}{1+f}. \end{cases}$$

If we now let s denote the generating function for the simple permutations of length at least 4 in $\mathcal{C}$, we find that

$$f = x + \frac{2f^2}{1+f} + s(f),$$

so if s is algebraic, a fortiori if s is polynomial, f is algebraic[7].

The following brief review of algebraic systems is a specialisation of the more general treatment in Stanley [33, Section 6.6]. Let $A = \{a_1, \ldots, a_n\}$ denote an alphabet. A *proper algebraic system* over $\mathbb{Q}[x_1, \ldots, x_m]$ is a set of equations $a_i = p_i(x_1, \ldots, x_m, a_1, \ldots, a_n)$ where each p_i is a polynomial with coefficients from $\mathbb{Q}$, has constant term 0, and contains no terms of the form ca_j where $c \in \mathbb{Q}$. The solution to such a system is a tuple $(f_1, \ldots, f_n)$ of formal power series from $\mathbb{Q}[[x_1, \ldots, x_m]]$ such that for all i, f_i is equal to $p_i(x_1, \ldots, x_m, a_1, \ldots, a_n)$ evaluated at $(a_1, \ldots, a_n) = (f_1, \ldots, f_n)$.

Theorem 3.5 (Stanley [33, Proposition 6.6.3 and Theorem 6.6.10])**.** *Every proper algebraic system $(p_1, \ldots, p_n)$ over $\mathbb{Q}[x_1, \ldots, x_m]$ has a unique solution $(f_1, \ldots, f_n)$. Moreover, each of these f_i's is algebraic over $\mathbb{Q}[x_1, \ldots, x_m]$.*

We need one final result before proving Theorem 1.1, which Albert and Atkinson derived from Higman's Theorem [20]:

Proposition 3.6 (Albert and Atkinson [2])**.** *Permutation classes with only finitely many simple permutations are finitely based.*

Theorem 1.1. *Let $\mathcal{C}$ be a permutation class containing only finitely many simple permutations, $\mathcal{P}$ a finite query-complete set of properties, and $\mathcal{Q} \subseteq \mathcal{P}$. The generating function for the set of permutations in $\mathcal{C}$ satisfying every property in $\mathcal{Q}$, i.e., $f_{\mathcal{Q}}$, is algebraic over $\mathbb{Q}[x]$.*

Proof. Let B denote the basis of $\mathcal{C}$, which is finite by Proposition 3.6. Lemma 2.1 shows that for every $\beta \in B$, the property $\mathrm{Av}(\beta)$ lies in a finite query-complete set. Thus the set $\{\mathrm{Av}(\beta) : \beta \in B\}$ is contained in a finite query-complete set, and we have

$$\mathcal{C} = \mathcal{W}(\mathcal{C})_{\{\mathrm{Av}(\beta) : \beta \in B\}}.$$

[7] In particular, note that the separable permutations correspond to $s = 0$; making this substitution leaves $f = x + 2f^2/(1+f)$, giving the large Schröder numbers.

Therefore it suffices to prove the theorem for wreath-closed classes. Furthermore, if $\mathcal{P}$ is query-complete then $\mathcal{P} \cup \{\not\oplus, \not\ominus\}$ is also query-complete, so we may assume without loss that $\not\oplus, \not\ominus \in \mathcal{P}$.

Let $\mathcal{P}(\pi)$ denote the set of properties in $\mathcal{P}$ satisfied by π and, to avoid inclusion-exclusion, let $g_{\mathcal{R}}$ denote the generating function for the set of $\pi \in \mathcal{C}$ with $\mathcal{P}(\pi) = \mathcal{R}$, so

$$f_{\mathcal{Q}} = \sum_{\mathcal{Q} \subseteq \mathcal{R} \subseteq \mathcal{P}} g_{\mathcal{R}}.$$

As $\mathcal{P}$ is query-complete, for each simple σ, $\mathcal{P}(\sigma[\alpha_1, \ldots, \alpha_m])$ is completely determined by σ and $\mathcal{P}(\alpha_1), \ldots, \mathcal{P}(\alpha_m)$. Thus for each simple σ of length m, there is a finite collection of m-tuples of sets of properties such that $\mathcal{P}(\sigma[\alpha_1, \ldots, \alpha_m]) = \mathcal{R}$ precisely if $(\mathcal{P}(\alpha_1), \ldots, \mathcal{P}(\alpha_m))$ lies in this collection. If $m \geq 4$ then Proposition 3.1 implies that the generating function for all inflations π of σ with $\mathcal{P}(\pi) = \mathcal{R}$ can be expressed nontrivially as a polynomial in $\{g_{\mathcal{S}} : \mathcal{S} \subseteq \mathcal{P}\}$ of degree m. If $m = 2$, suppose $\sigma = 12$ without loss. By Proposition 3.2, all inflations of 12 have a unique decomposition as $12[\alpha_1, \alpha_2]$ where $\alpha_1 \in \not\oplus$. Thus the generating function for inflations π of 12 with $\mathcal{P}(\pi) = \mathcal{R}$ can be expressed as a sum of terms of the form $g_{\mathcal{S}} g_{\mathcal{T}}$ where $\not\oplus \in \mathcal{S}$.

Therefore $g_{\mathcal{R}}$ can be expressed as a polynomial in x (depending on whether $\mathcal{P}(1) = \mathcal{R}$) and $\{g_{\mathcal{S}} : \mathcal{S} \subseteq \mathcal{P}\}$. Moreover, these polynomials have no constant terms and no terms of the form $c g_{\mathcal{S}}$ for constant $c \neq 0$. Thus they form a proper algebraic system, so Theorem 3.5 implies that each $g_{\mathcal{S}}$ is algebraic. □

Corollary 1.4 — with the exception of the involution case, discussed in Section 5 — now follows from Theorem 1.1 and the collection of query-complete sets in Section 2.

4. EXAMPLES

While we have already shown how to enumerate separable permutations in Footnote 7, here we use the approach of Theorem 1.1.

Example 4.1: Separable permutations. With the notation from the proof of Theorem 1.1, we have that for the separable permutations:

$$\begin{cases} g_{\not\oplus,\not\ominus} & = & x, \\ g_{\not\oplus} & = & (g_{\not\oplus,\not\ominus} + g_{\not\ominus})(g_{\not\oplus,\not\ominus} + g_{\not\oplus} + g_{\not\ominus}), \\ g_{\not\ominus} & = & (g_{\not\oplus,\not\ominus} + g_{\not\oplus})(g_{\not\oplus,\not\ominus} + g_{\not\oplus} + g_{\not\ominus}), \end{cases}$$

where our universe of properties $\mathcal{P}$ is $\{\not\oplus, \not\ominus\}$. We are interested in $f = g_{\not\oplus,\not\ominus} + g_{\not\oplus} + g_{\not\ominus}$. By summing the three equalities above and simplifying one obtains $f = x + (x + f)f$, which leads, reassuringly, to the generating function for the large Schröder numbers,

$$f = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2}.$$

This system does not change dramatically when another simple permutation is introduced, as shown by the next example.

Example 4.2: The wreath closure of 1, 12, 21, and 2413. Here we again take $\mathcal{P} = \{\not\oplus, \not\ominus\}$ and the system is

$$\begin{cases} g_{\not\oplus,\not\ominus} & = & x + (g_{\not\oplus,\not\ominus} + g_{\not\oplus} + g_{\not\ominus})^4, \\ g_{\not\oplus} & = & (g_{\not\oplus,\not\ominus} + g_{\not\ominus})(g_{\not\oplus,\not\ominus} + g_{\not\oplus} + g_{\not\ominus}), \\ g_{\not\ominus} & = & (g_{\not\oplus,\not\ominus} + g_{\not\oplus})(g_{\not\oplus,\not\ominus} + g_{\not\oplus} + g_{\not\ominus}). \end{cases}$$

The generating function for this class, $f = g_{\not\oplus,\not\ominus} + g_{\not\oplus} + g_{\not\ominus}$, satisfies

$$f^5 + f^4 + f^2 + (x-1)f + x = 0.$$

Example 4.3: $\mathbf{Av(132)}$. The wreath closure of $\mathrm{Av}(132)$ is the class of separable permutations, so to enumerate $\mathrm{Av}(132)$ we need to refine Example 4.1. While Proposition 2.1 shows that $\{\mathrm{Av}(1), \mathrm{Av}(12), \mathrm{Av}(21), \mathrm{Av}(132)\}$ is query-complete, we remarked at the beginning of Section 3 that setting $\mathcal{P} = \{\not\oplus, \not\ominus, \mathrm{Av}(21), \mathrm{Av}(132)\}$ will suffice. Our system is then

$$\begin{cases} g_{\not\oplus,\not\ominus,\mathrm{Av}(21)} & = & x, \\ g_{\not\ominus,\mathrm{Av}(21)} & = & g_{\not\oplus,\not\ominus,\mathrm{Av}(21)}(g_{\not\oplus,\not\ominus,\mathrm{Av}(21)} + g_{\not\ominus,\mathrm{Av}(21)}), \\ g_{\not\oplus} & = & (g_{\not\oplus,\not\ominus,\mathrm{Av}(21)} + g_{\not\ominus,\mathrm{Av}(21)} + g_{\not\ominus})(g_{\not\oplus,\not\ominus,\mathrm{Av}(21)} + g_{\not\ominus,\mathrm{Av}(21)} + g_{\not\oplus} + g_{\not\ominus}), \\ g_{\not\ominus} & = & g_{\not\oplus}(g_{\not\oplus,\not\ominus,\mathrm{Av}(21)} + g_{\not\ominus,\mathrm{Av}(21)}). \end{cases}$$

(As we are only interested in 132-avoiding permutations we have suppressed the subscript $\mathrm{Av}(132)$, which would otherwise be present in all these terms.) Setting

$$f = g_{\not\oplus,\not\ominus,\mathrm{Av}(21)} + g_{\not\ominus,\mathrm{Av}(21)} + g_{\not\oplus} + g_{\not\ominus}$$

and solving yields

$$f = \frac{1 - 2x - \sqrt{1-4x}}{2x},$$

the generating function for the Catalan numbers, as expected.

Example 4.4: $\mathbf{Av(2413, 3142, 2143)}$. Here we take $\mathcal{P} = \{\not\oplus, \not\ominus, \mathrm{Av}(21), \mathrm{Av}(2143)\}$ and our system is

$$\begin{cases} g_{\not\oplus,\not\ominus,\mathrm{Av}(21)} & = & x, \\ g_{\not\ominus,\mathrm{Av}(21)} & = & g_{\not\oplus,\not\ominus,\mathrm{Av}(21)}(g_{\not\oplus,\not\ominus,\mathrm{Av}(21)} + g_{\not\ominus,\mathrm{Av}(21)}), \\ g_{\not\oplus} & = & (g_{\not\oplus,\not\ominus,\mathrm{Av}(21)} + g_{\not\ominus,\mathrm{Av}(21)} + g_{\not\ominus})(g_{\not\oplus,\not\ominus,\mathrm{Av}(21)} + g_{\not\ominus,\mathrm{Av}(21)} + g_{\not\oplus} + g_{\not\ominus}), \\ g_{\not\ominus} & = & g_{\not\oplus,\not\ominus,\mathrm{Av}(21)}(g_{\not\oplus} + g_{\not\ominus}) + g_{\not\oplus}(g_{\not\oplus,\not\ominus,\mathrm{Av}(21)} + g_{\not\ominus,\mathrm{Av}(21)}), \end{cases}$$

where here we have suppressed the $\mathrm{Av}(2143)$ subscript. This gives the generating function

$$\frac{1 - 3x + 2x^2 - \sqrt{1 - 6x + 5x^2}}{2x(2-x)},$$

and thus the number of permutations of length n in this class is $\sum \binom{n}{k} F_{n-k}$ where F_n denotes the nth term in Fine's sequence[8].

Example 4.5: Alternating separable permutations. Lemma 2.2 shows that we need to introduce the properties AL (alternating permutations), BR (permutations beginning with a rise), ER (permutations ending with a rise), and $\{1\}$. In the separable case $\{1\} = \not\oplus \cap \not\ominus$ so we take $\mathcal{P} = \{\not\oplus, \not\ominus, BR, ER, AL\}$, and as AL occurs in each of the terms of our system we suppress it. We then have

$$\begin{cases} g_{\not\oplus,\not\ominus} & = & x, \\ g_{\not\oplus} & = & (g_{\not\oplus,\not\ominus} + g_{\not\ominus,ER})(g_{\not\oplus,\not\ominus} + g_{\not\oplus,BR} + g_{\not\ominus,BR}), \\ g_{\not\oplus,BR} & = & g_{\not\ominus,BR,ER}(g_{\not\oplus,\not\ominus} + g_{\not\oplus,BR} + g_{\not\ominus,BR}), \\ g_{\not\oplus,ER} & = & (g_{\not\oplus,\not\ominus} + g_{\not\ominus,ER})(g_{\not\oplus,BR,ER} + g_{\not\ominus,BR,ER}), \\ g_{\not\oplus,BR,ER} & = & g_{\not\ominus,BR,ER}(g_{\not\oplus,BR,ER} + g_{\not\ominus,BR,ER}), \\ g_{\not\ominus} & = & g_{\not\oplus}(g_{\not\oplus} + g_{\not\ominus}), \\ g_{\not\ominus,BR} & = & (g_{\not\oplus,\not\ominus} + g_{\not\oplus,BR})(g_{\not\oplus} + g_{\not\ominus}), \\ g_{\not\ominus,ER} & = & g_{\not\oplus}(g_{\not\oplus,\not\ominus} + g_{\not\oplus,ER} + g_{\not\ominus,ER}), \\ g_{\not\ominus,BR,ER} & = & (g_{\not\oplus,\not\ominus} + g_{\not\oplus,BR})(g_{\not\oplus,\not\ominus} + g_{\not\oplus,ER} + g_{\not\ominus,ER}). \end{cases}$$

The generating function for these permutations satisfies

$$f^3 - (2x^2 - 5x + 4)f^2 - (4x^3 + x^2 - 8x)f - (2x^4 + 5x^3 + 4x^2) = 0.$$

5. INVOLUTIONS

Unfortunately, involutionhood lies just outside the scope of our query-complete-property machinery: letting I denote the set of involutions we have that $12[\alpha_1, \alpha_2] \in I \iff \alpha_1, \alpha_2 \in I$, but when is $21[\alpha_1, \alpha_2] \in I$?

We begin by considering the effect of inversion on the substitution decomposition. First observe that

$$(\sigma[\alpha_1, \ldots, \alpha_m])^{-1} = \sigma^{-1}[\alpha^{-1}_{\sigma^{-1}(1)}, \ldots, \alpha^{-1}_{\sigma^{-1}(m)}].$$

Recalling Proposition 1.2 ("every permutation is the inflation of a unique simple permutation"), we have that if π is an involution then it must be the inflation of a simple involution. By Proposition 3.1 we then obtain the following:

Proposition 5.1. *If $\pi = \sigma[\alpha_1, \ldots, \alpha_m]$ is an involution and $\sigma \neq 21$ is a simple permutation then σ is an involution and $\alpha_i = \alpha^{-1}_{\sigma^{-1}(i)} = \alpha^{-1}_{\sigma(i)}$ for all $i \in [m]$.*

The case $\sigma = 21$ must be handled separately but is not any more difficult.

Proposition 5.2. *The involutions that are inflations of 21 are precisely those of the form*

[8] *Fine's sequence* is defined by $2F_n + F_{n-1} = C_n$ for $n \geq 1$, where C_n denotes the nth Catalan number.

- $21[\alpha_1, \alpha_2]$ *for skew indecomposable* α_1 *and* α_2 *with* $\alpha_1 = \alpha_2^{-1}$, *and*
- $321[\alpha_1, \alpha_2, \alpha_3]$, *where* α_1 *and* α_3 *are skew indecomposable,* $\alpha_1 = \alpha_3^{-1}$, *and* α_2 *is an involution.*

Define the *inverse* of the property P by $P^{-1} = \{\pi^{-1} : \pi \in P\}$, and for a set of properties $\mathcal{P}$, $\mathcal{P}^{-1} = \{P^{-1} : P \in \mathcal{P}\}$.

Theorem 5.3. *Let $\mathcal{C}$ be a permutation class containing only finitely many simple permutations, $\mathcal{P}$ a finite query-complete set of properties, and $\mathcal{Q} \subseteq \mathcal{P}$. The generating function for the set of involutions in $\mathcal{C}$ satisfying every property in $\mathcal{Q}$ is algebraic over $\mathbb{Q}[x]$.*

Proof. We assume (without loss) both that $\not\emptyset, \emptyset \in \mathcal{P}$ and that $\mathcal{P} = \mathcal{P}^{-1}$. As in the proof of Theorem 1.1, let $\mathcal{P}(\pi)$ denote the set of properties in $\mathcal{P}$ satisfied by π and $g_{\mathcal{R}}$ denote the generating function for the set of $\pi \in \mathcal{C}$ with $\mathcal{P}(\pi) = \mathcal{R}$. Also let $h_{\mathcal{R}}$ denote the generating function for the set of involutions $\pi \in \mathcal{C}$ with $\mathcal{P}(\pi) = \mathcal{R}$. It suffices to show that each $h_{\mathcal{R}}$ is algebraic over $\mathbb{Q}[x]$.

As Propositions 5.1 and 5.2 indicate, we need to count pairs (α, α^{-1}) where α and α^{-1} satisfy certain sets of properties. To this end define

$$p_{\mathcal{R}} = \sum_{\substack{\alpha \in \mathcal{C} \\ \mathcal{P}(\alpha) = \mathcal{R}}} x^{|\alpha| + |\alpha^{-1}|}.$$

Note that if $\mathcal{P}(\alpha) = \mathcal{R}$ then $\mathcal{P}(\alpha^{-1}) = \mathcal{R}^{-1}$ because $\mathcal{P} = \mathcal{P}^{-1}$, and thus $p_{\mathcal{R}} = g_{\mathcal{R}}(x^2)$.

Now take σ to be a simple permutation. We need to compute the contribution to $h_{\mathcal{R}}$ of inflations of σ. If σ is not an involution, Proposition 5.1 shows that this contribution is 0. Otherwise since $\mathcal{P}$ is query-complete, $\mathcal{P}(\sigma[\alpha_1, \ldots, \alpha_m]) = \mathcal{R}$ if and only if $(\mathcal{P}(\alpha_1), \ldots, \mathcal{P}(\alpha_m))$ lies in a certain collection of m-tuples of sets of properties. Choose one of these m-tuples, say $(\mathcal{R}_1, \ldots, \mathcal{R}_m)$, and suppose first that $m = |\sigma| \geq 4$. It suffices to calculate the contribution of involutions of the form $\sigma[\alpha_1, \ldots, \alpha_m]$ with $\mathcal{P}(\alpha_i) = \mathcal{R}_i$ for all $i \in [m]$. If there is some $j \in [m]$ for which $\mathcal{R}_j \neq \mathcal{R}_{\sigma(j)}^{-1}$ then this contribution is 0 by Proposition 5.1. Otherwise the contribution is a single term in which each fixed point j corresponds to an $h_{\mathcal{R}_j}$ factor and each non-fixed-point pair $(j, \sigma(j))$ corresponds to a $p_{\mathcal{R}_j}$ factor. A similar analysis of inflations of 12 and 21 — in the latter case appealing to Proposition 5.2 — allows us to compute their contributions.

Therefore each $h_{\mathcal{R}}$ can be expressed nontrivially as a polynomial in x, $\{h_{\mathcal{S}} : \mathcal{S} \subseteq \mathcal{P}\}$, and $\{p_{\mathcal{S}} : \mathcal{S} \subseteq \mathcal{P}\}$. Viewing x and $\{p_{\mathcal{S}} : \mathcal{S} \subseteq \mathcal{P}\}$ as variables, Theorem 3.5 implies that each $h_{\mathcal{R}}$ is algebraic over $\mathbb{Q}[x, \{p_{\mathcal{S}} : \mathcal{S} \subseteq \mathcal{P}\}]$. Furthermore, $p_{\mathcal{S}} = g_{\mathcal{S}}(x^2)$, so $\mathbb{Q}(x, \{p_{\mathcal{S}} : \mathcal{S} \subseteq \mathcal{P}\})$ is an algebraic extension of $\mathbb{Q}(x)$ by Theorem 1.1, proving the theorem. □

One could adapt the proof of Theorem 5.3 to count the permutations in $\mathcal{C}$ that are invariant under other symmetries. For example, the permutations invariant under the composition of reverse and complement studied by Guibert and Pergola [19]. Egge [10] considers the enumeration of restricted permutations invariant under other symmetries.

Example 5.4: Separable involutions. We take $\mathcal{P} = \{\not\oplus, \not\ominus\}$. Using the notation from the proof of Theorem 5.3, we wish to find $f = h_{\not\oplus,\not\ominus} + h_{\not\oplus} + h_{\not\ominus}$. These generating functions are related to each other and to the p generating functions by

$$\begin{cases} h_{\not\oplus,\not\ominus} & = & x, \\ h_{\not\oplus} & = & (p_{\not\oplus,\not\ominus} + p_{\not\ominus}) + (p_{\not\oplus,\not\ominus} + p_{\not\ominus})(h_{\not\oplus,\not\ominus} + h_{\not\oplus} + h_{\not\ominus}), \\ h_{\not\ominus} & = & (h_{\not\oplus,\not\ominus} + h_{\not\oplus})(h_{\not\oplus,\not\ominus} + h_{\not\oplus} + h_{\not\ominus}). \end{cases}$$

From Example 4.1 it can be computed that

$$\begin{array}{rcl} p_{\not\oplus,\not\ominus} - x^2 & = & 0, \\ 2p_{\not\oplus}^2 + (3x^2-1)p_{\not\oplus} + x^4 & = & 0, \\ 2p_{\not\ominus}^2 + (3x^2-1)p_{\not\ominus} + x^4 & = & 0. \end{array}$$

Combining these with the system above and solving as usual shows that

$$x^2f^4 + (x^3+3x^2+x-1)f^3 + (3x^3+6x^2-x)f^2 + (3x^3+7x^2-x-1)f + x^3 + 3x^2 + x = 0.$$

6. CYCLIC CLOSURES

In this section we present an application of Theorem 1.1 which differs in flavour from our previous uses. The permutation τ is said to be a *cyclic rotation* (or simply, rotation) of the permutation π, both of length n, if there is an $i \in [n]$ for which $\tau = \pi(i+1)\dots\pi(n)\pi(1)\dots\pi(i)$. Given a permutation class $\mathcal{C}$, its *cyclic closure*, $\mathrm{cc}(\mathcal{C})$, consists of all rotations of members of $\mathcal{C}$. This operation was first studied by the Otago group [1], who proved several basis and enumeration results. The main result of this section, Theorem 6.2, shows that the cyclic closure of a class with finitely many simple permutations has an algebraic generating function.

The cyclic closure of the class $\mathcal{C}$ can be partitioned into orbits of permutations under rotation. As the orbit of a permutation of length n has precisely n elements, to enumerate a cyclic closure it suffices to count orbits. We do this by distinguishing one permutation per orbit and then counting these permutations. For us, a *distinguished* member of $\mathrm{cc}(\mathcal{C})$ is a permutation π that satisfies:

(1) $\pi \in \mathcal{C}$ (this can clearly be achieved, because every orbit in $\mathrm{cc}(\mathcal{C})$ contains at least one element of $\mathcal{C}$) and

(2) among all permutations in its orbit satisfying (1), π is the one in which the entry 1 lies furthest to the left.

For example, one orbit in $\mathrm{cc}(\mathrm{Av}(132))$ is

$$12534, 41253, 34125, 53412, 25341.$$

Only two of these permutations avoid 132, 34125 and 53412. Since the entry 1 lies further to the left in 34125, this is the distinguished permutation of its orbit.

Our goal is to show that the property of distinction lies in a finite query-complete set of properties. We begin by offering a different viewpoint in which instead of rotating permutations we divide them into two parts. A *divided permutation* is a permutation equipped with a divider $|$, i.e., $\pi_1|\pi_2$, and we refer to $\pi_1|\pi_2$ as a *division* of the concatenation $\pi_1\pi_2$. We say that the divided permutation $\sigma_1|\sigma_2$ is contained in the divided permutation $\pi_1|\pi_2$ if $\pi_1\pi_2$ contains a subsequence order isomorphic to $\sigma_1\sigma_2$ in which the entries corresponding to σ_1 come from π_1 and the entries corresponding to σ_2 come from π_2. For example, $513|42$ contains $32|1$ because of the subsequence 532, but $32|1$ is not contained in $51|342$.

Suppose now that we are given a permutation $\pi \in \mathcal{C} = \mathrm{Av}(B)$ and we wish to decide if π is a distinguished member of $\mathrm{cc}(\mathcal{C})$. According to (2) above, we need to check all rotations of π in which the 1 lies further to the left. Instead, let us consider all divisions $\pi_1|\pi_2$ of π in which π_1 is nonempty and π_2 contains the entry 1, thinking of such a division as corresponding to the rotation $\pi_2\pi_1$. For π to be distinguished, each of these divisions must contain $\beta_2|\beta_1$ for some $\beta_1\beta_2 \in B$, because that will imply that the corresponding rotation contains $\beta_1\beta_2$ and thus fails to lie in $\mathcal{C}$.

For a set of divided permutations Δ, let us therefore define the property $DP_1(\Delta)$ to consist of all permutations π for which every division $\pi_1|\pi_2$ where π_1 is nonempty and the 1 lies in π_2 contains at least one of the divided permutations in Δ. Our set of distinguished permutations for $\mathrm{cc}(\mathcal{C})$ will then consist of those permutations from $\mathcal{C}$ which satisfy

$$DP_1(\{\beta_2|\beta_1 : \beta_1\beta_2 \in B\}).$$

We also need a similar family: $DP(\Delta)$ consists of all permutations π for which every division $\pi_1|\pi_2$ of π in which π_1 is nonempty contains at least one of the divided permutations in Δ. (Note that we allow π_2 to be empty.)

Lemma 6.1. *For any finite set B of permutations, the property $DP_1(\{\beta_2|\beta_1 : \beta_1\beta_2 \in B\})$ lies in a finite query-complete set of properties.*

Proof. The finite query-complete set we take consists of

$$\{\mathrm{Av}(\delta) : \delta \leq \beta \text{ for some } \beta \in B\}$$

and the properties $DP(\Delta)$ and $DP_1(\Delta)$ for all $\Delta \subset \{\delta_2|\delta_1 : \delta_1\delta_2 \leq \beta \text{ for some } \beta \in B\}$.

Let $\pi = \sigma[\alpha_1, \ldots, \alpha_m]$. Propositions 1.3 and 2.1 show that the Av properties form a query-complete set, so it suffices to prove that membership in the DP and DP_1 can be decided based on σ and which of these properties each α_i satisfies. Since these properties are very similar, we consider only the $DP_1(\Delta)$ case.

Suppose that $\sigma(\ell) = 1$, so that the entry 1 in π occurs in its $\sigma(\ell)$ interval. First, for each $k < \ell$, we need to consider divisions of π which slice its $\sigma(k)$ interval (or slice between this interval and the next). As in the proof of Proposition 2.1 we consider lenient inflations (inflations in which intervals are allowed to be empty), although we now insist that the divider occur in the kth interval of the lenient inflations (we allow that interval to contain the divider alone). List all such lenient inflations of all divided permutations in Δ as

$$\sigma[\gamma_1^{(1)}, \ldots, \gamma_m^{(1)}], \ldots, \sigma[\gamma_1^{(t)}, \ldots, \gamma_m^{(t)}].$$

We need to determine whether every division of π which slices its $\sigma(k)$ interval contains one of these lenient inflations. If for some $s \in [t]$ and $j \neq k$, α_j does not contain $\gamma_j^{(s)}$ (which can be determined from the Av properties), then none of these divisions of π can contain that lenient inflation. Remove these infeasible inflations from the list, leaving

$$\sigma[\gamma_1^{(u_1)}, \ldots, \gamma_m^{(u_1)}], \ldots, \sigma[\gamma_1^{(u_v)}, \ldots, \gamma_m^{(u_v)}].$$

Now a division of π slicing its $\sigma(k)$ interval contains the ith lenient inflation in this list if and only if $\gamma_k^{(u_i)}$ is either a lone divider or is contained (as a divided permutation) in the resulting, divided α_k. Thus every division of π which slices its $\sigma(k)$ interval contains a divided permutation from Δ if and only if

$$\alpha_k \in DP(\{\gamma_k^{(u_1)}, \ldots, \gamma_k^{(u_v)}\}),$$

and this property is in our set of properties. The analysis for divisions of π which slice the $\sigma(\ell)$ interval (the block containing the entry 1) is identical, except that DP is replaced by DP_1. ☐

Theorem 6.2. *If a permutation class $\mathcal{C}$ contains only finitely many simple permutations then its cyclic closure $\mathrm{cc}(\mathcal{C})$ has an algebraic generating function over $\mathbb{Q}[x]$.*

Proof. Let $\mathcal{C} = \mathrm{Av}(B)$ contain only finitely many simple permutations, so by Proposition 3.6, B is finite. Lemma 6.1 the shows that the property $DP_1(\{\beta_2|\beta_1 : \beta_1\beta_2 \in B\})$ lies in a finitely query-complete set. Thus the distinguished permutations, which are the permutations in $\mathcal{C}$ that satisfy this property, have an algebraic generating function by Theorem 1.1. Call this generating function f. Since every orbit of length n permutations in $\mathrm{cc}(C)$ contains n elements, precisely one of which is distinguished, the generating function for $\mathrm{cc}(\mathcal{C})$ is $xf'(x)$, which is also algebraic. ☐

We conclude the section with an abridged example.

Example 6.3: The cyclic closure of $\mathbf{Av(132)}$. The distinguished elements for $\mathrm{cc}(\mathrm{Av}(132))$ are those that lie in $\mathrm{Av}(132)$ and satisfy

$$DP_1(\{\beta_2|\beta_1 : \beta_1\beta_2 = 132\}) = DP_1(132|, 32|1, 2|13, |132).$$

If any division of a permutation contains $132|$ or $|132$ then the permutation itself contains 132; since we are only counting 132-avoiding permutations, we may write the generating function for the distinguished elements as $f_{DP_1(32|1,2|13)}$, where $f_{\mathcal{Q}}$ denotes the generating function for the permutations in $\mathrm{Av}(132)$ which satisfy every property in $\mathcal{Q}$ but may satisfy additional properties. In the other examples we have given the complete system of g generating functions. Owing to the number of properties involved and the labour necessary for their specification, here we only describe how to compute two of the f generating functions.

Let us begin with the $f_{\not\emptyset, DP_1(32|1,2|13)}$ term. Since our only simple permutations are 1, 12, 21, the $\oplus$-indecomposable permutations are 1 and those that can be expressed uniquely as $21[\alpha_1, \alpha_2]$ where $\alpha_1 \in \not\emptyset$. First consider divisions of $21[\alpha_1, \alpha_2]$ which slice α_1; for these to contain either $32|1$ or $2|13$, the divided α_1 must contain either $21|$, which can be extended to $32|1$ by including an entry of α_2, or $2|13$. All such permutations must contain 21, so they are counted by $f_{\not\emptyset, DP(21|,2|13)} - f_{\not\emptyset,\mathrm{Av}(21),DP(21|,2|13)}$. Now observe that the divisions which slice α_2 before its entry 1 necessarily contain a copy of $32|1$ where the '3' comes from α_1 and the '2' comes from an entry of α_2 preceding 1 (if there is no such entry, then none of these divisions need checking), and so every 132-avoiding permutation may serve as α_2. Thus we have

$$f_{\not\emptyset, DP_1(32|1,2|13)} = x + \left(f_{\not\emptyset, DP(21|,2|13)} - f_{\not\emptyset,\mathrm{Av}(21),DP(21|,2|13)}\right) f.$$

This leaves us to determine $f_{\not\emptyset, DP(21|,2|13)}$. These permutations (except for 1) can be written uniquely as $\pi = 12[\alpha_1, \alpha_2]$ where $\alpha_1 \in \not\emptyset$ and as they avoid 132 we have $\alpha_2 \in \mathrm{Av}(21)$. The divisions slicing α_1 must create $21|$ or $2|13$ patterns in π, which will occur if and only if $\alpha_1 \in DP(21|,2|1)$. This rules out $\alpha_1 = 1$, so these permutations are counted by $f_{\not\emptyset, DP(21|,2|1)} - x$. Because $\alpha \in DP(21|,2|1)$, α_1 must contain 21, and thus all divisions which slice α_2 will contain $21|$. Therefore the only restriction on α_2 is that it must avoid 21, giving the equation

$$f_{\not\emptyset, DP(21|,2|13)} = x + \left(f_{\not\emptyset, DP(21|,2|1)} - x\right) f_{\mathrm{Av}(21)}.$$

Similar reasoning allows one to compute the entire system, which leads to the solution

$$f_{DP_1(32|1,2|13)} = \frac{(1-2x)(1-2x-\sqrt{1-4x})}{2x(1-x)}.$$

From this we find that the generating function for $\mathrm{cc}(\mathrm{Av}(132))$ is

$$x f'_{DP_1(32|1,2|13)} = \frac{1-4x+4x^2-4x^3-(1-2x)\sqrt{1-4x}}{2x(1-x)^2\sqrt{1-4x}},$$

which agrees with the results of Albert et al. [1].

7. APPLICABILITY AND APPLICATION

With the results of the paper now established, we conclude by discussing their use.

Determining if these methods apply. As these techniques apply only to permutation classes with finitely many simple permutations, it would be useful to be able to determine whether a permutation class contains finitely many simple permutations. This can be done:

Theorem 7.1 (Brignall, Ruškuc, and Vatter [7])**.** *It is decidable whether a permutation class given by a finite basis contains only finitely many simple permutations.*

Finding the simple permutations. Thus far we have tacitly assumed that the set of simple permutations in our class is known. Since classes are often specified by their bases, this set of simple permutations must first be computed. Assuming that this set is finite, it can be computed via a result of Schmerl and Trotter. While we state only the permutation case (a proof of this case is also given by Murphy [28]), their result covers all irreflexive binary relational structures. See Ehrenfeucht and McConnell [14] for a version of this theorem for certain other structures.

Theorem 7.2 (Schmerl and Trotter [30])**.** *Every simple permutation of length $n \geq 2$ contains a simple permutation of length $n-1$ or $n-2$.*

For example, the number of simple permutations in $\mathrm{Av}(1324, 2143, 4231)$ of lengths 1 to 7 is $1, 2, 0, 2, 4, 0, 0$. Because there are no simple permutations of length 6 or 7 in this class, Theorem 7.2 ensures that it contains no longer simple permutations.

Computing wreath-closures. Conversely, given a finite set of simple permutations, one may ask for the basis of its wreath closure. Theorem 7.2 gives a method for its computation:

Proposition 7.3. *If the longest simple permutations in $\mathcal{C}$ have length k then the basis elements of $\mathcal{W}(\mathcal{C})$ have length at most $k+2$.*

Proof. The basis of $\mathcal{W}(\mathcal{C})$ is easily seen to consist of the minimal (under the pattern-containment order) simple permutations not contained in $\mathcal{C}$ (cf. Proposition 3.3). Let π be such a permutation of length n. Theorem 7.2 shows that π contains a simple permutation σ of length $n-1$ or $n-2$. If $n \geq k+3$, then $\sigma \notin \mathcal{C}$, so $\sigma \notin \mathcal{W}(\mathcal{C})$ and thus π cannot lie in the basis of $\mathcal{W}(\mathcal{C})$. □

Using this proposition it can be computed that the wreath closure of 1, 12, 21, and 2413 considered in Example 4.2 is $\mathrm{Av}(3142, 25314, 246135, 362514)$.

Other reasons for algebraicity. Having finitely many simple permutations is a sufficient condition for a class to possess an algebraic generating function, but it is by no means necessary. Consider $\mathrm{Av}(123)$, which, like $\mathrm{Av}(132)$, is enumerated by the Catalan numbers. However, $\mathrm{Av}(123)$ contains the infinite sequence of simple permutations $2n-1, 2n-3, \ldots, 3, 1, 2n, 2n-2, \ldots, 4, 2$ (one such permutation is plotted in Figure 1). Indeed, every class of the form $\mathrm{Av}(\beta)$ where $|\beta| \geq 4$ contains either this infinite family or a symmetry of it.

Derangements. Notably absent from our list of finite query-complete sets in Section 2 are derangements, despite the fact that the 132-avoiding derangements are counted by Fine's sequence (Robertson, Saracino, and Zeilberger [29]), which has an algebraic generating function. To see that the set of derangements does not lie in a finite query-complete set of properties, for $\alpha \in S_n$ define $D(\alpha) = \{\alpha(i) - i : i \in [n]\}$. Then $21[12 \cdots j, \alpha]$ is a derangement if and only if $j \notin D(\alpha)$. This shows that α_1 and α_2 must lie in different sets of properties whenever $D(\alpha_1) \cap \mathbb{N} \neq D(\alpha_2) \cap \mathbb{N}$, implying that the set of derangements can only lie in an infinite query-complete set of properties.

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