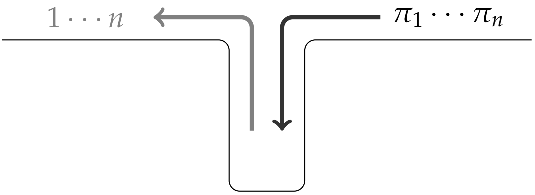
Generating functions of permutation classes

Robert Brignall

3rd April 2025

The Open University

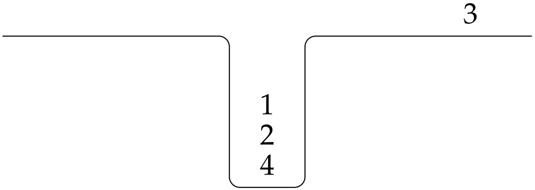


$1 \cdots n$
$\pi_1 \cdots \pi_n$

4

2

4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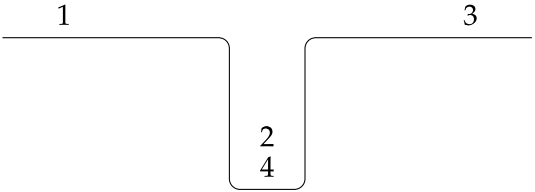
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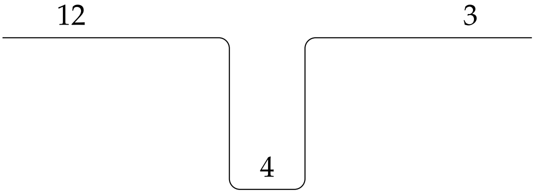
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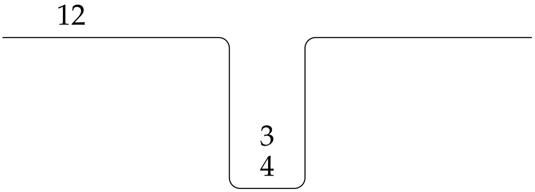
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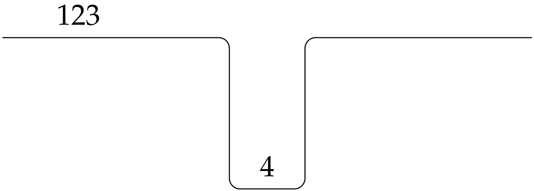
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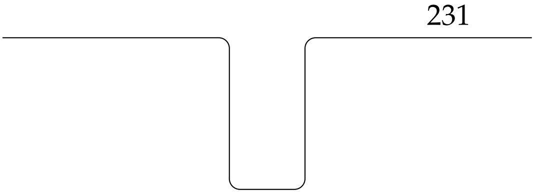
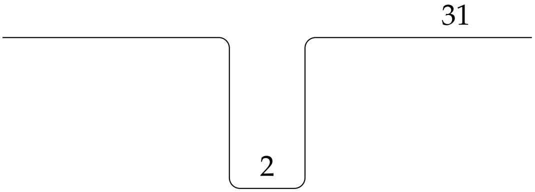
4

1234


231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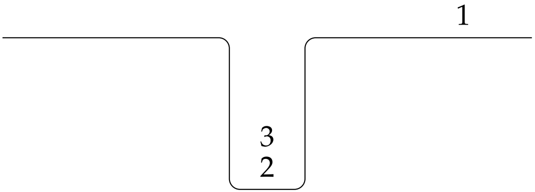
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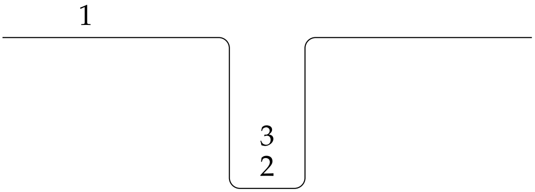
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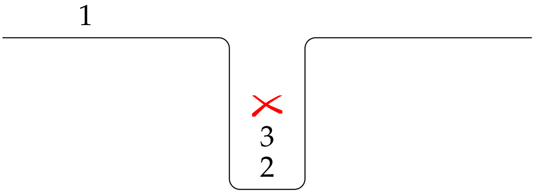
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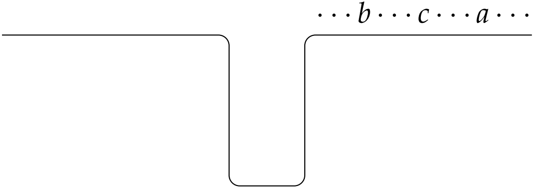
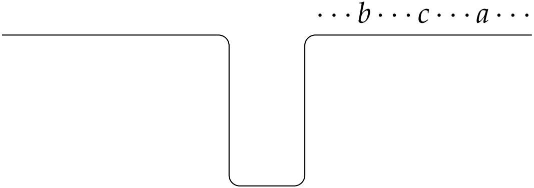


Any sequence $\cdots b \cdots c \cdots a \cdots$ where $a < b < c$ will fail.

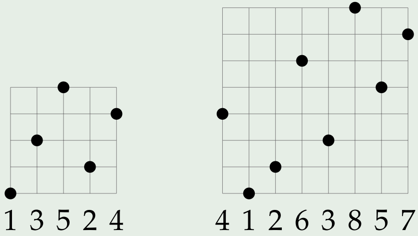


Any sequence $\cdots b \cdots c \cdots a \cdots$ where $a < b < c$ will fail.

A permutation that contains a copy of '231' can't be stack sorted.

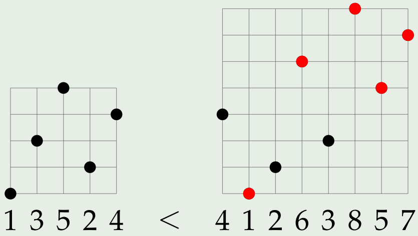
§1 Containment 101 and 102

Permutation containment 101



- Permutations in one-line notation: $\pi = \pi(1) \cdots \pi(n)$
- Pattern containment: $\sigma \leq \pi$ if there exists a subsequence of $\pi(1) \cdots \pi(n)$ with the same relative ordering as σ.
- Containment is a partial order.
- Conversely, π avoids σ if $\sigma \not\leq \pi$.

Permutation containment 101



- Permutations in one-line notation: $\pi = \pi(1) \cdots \pi(n)$
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Permutation containment 102

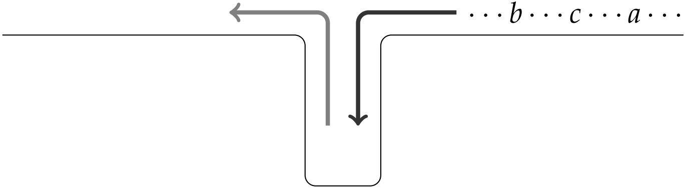
Permutation class: a hereditary ('downwards closed') collection $\mathcal{C}$, i.e.

$$\pi \in \mathcal{C} \text{ and } \sigma \leq \pi \text{ implies } \sigma \in \mathcal{C}.$$

Basis of the class: A unique minimal avoidance set that characterises the class precisely,

$$\mathcal{C} = \text{Av}(\beta_1, \ldots, \beta_k) = \{\text{permutations } \pi : \beta_i \not\leq \pi \text{ for all } i\}.$$

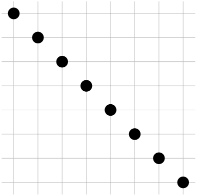
N.B. The basis need not be finite. When it is, then the class is *finitely based*.



A permutation that contains a copy of 231 can't be stack sorted.

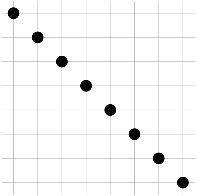
In fact: The stack-sortable permutations are precisely the class Av(231).

Examples



$\text{Av}(12) = \{1, 21, 321, \ldots\}$ has 1 permutation of each length.

Examples



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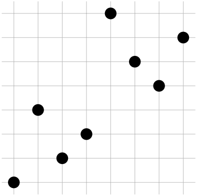
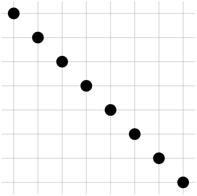


$\mathrm{Av}(231)$ has $1, 2, 5, 14, 42, \ldots$ of lengths $n = 1, 2, 3, 4, 5, \ldots$.

Examples



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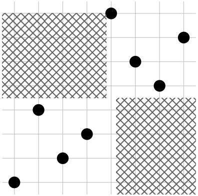

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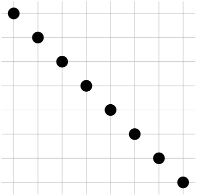

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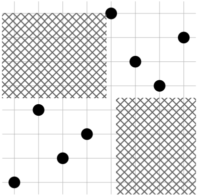

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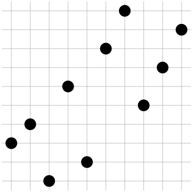

$\mathrm{Av}(321)$ has $1, 2, 5, 14, 42, \ldots$ of lengths $n = 1, 2, 3, 4, 5, \ldots$.

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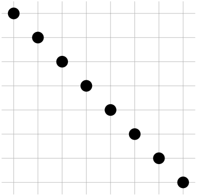
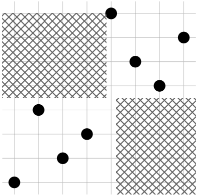
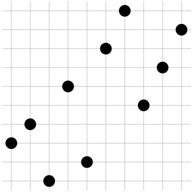


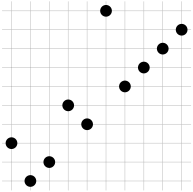
Av(12) = {1, 21, 321, . . . } has 1 permutation of each length.



Av(231) has 1, 2, 5, 14, 42, . . . of lengths $n = 1, 2, 3, 4, 5, \ldots$.



Av(321) has 1, 2, 5, 14, 42, . . . of lengths $n = 1, 2, 3, 4, 5, \ldots$.



Av(231, 321) has 2^{n-1} of length n.

Connections

Combinatorics/model theory Part of broader study of combinatorial structures (includes graphs)

Algebraic geometry A Schubert variety X_π is smooth $\iff$ $\pi \in \mathrm{Av}(3412, 4231)$ [Lakshmibai & Sandhya, 1990]

Statistical mechanics e.g. connections with *partially asymmetric simple exclusion processes* (PASEPs) [Corteel & Williams, 2007]

Computer science e.g. sorting algorithms and pattern matching

Evolutionary biology e.g. distances between gene sequences

§2 Enumeration

Counting...

...precisely

Generating function for a class $\mathcal{C}$ is the formal power series

$$f_{\mathcal{C}}(z) = \sum_{\pi \in \mathcal{C}} z^{|\pi|} = \sum_{n=1}^{\infty} |\mathcal{C}_n| z^n,$$

where $\mathcal{C}_n = \{\pi \in \mathcal{C} : |\pi| = n\}$.

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where $\mathcal{C}_n = \{\pi \in \mathcal{C} : |\pi| = n\}$.

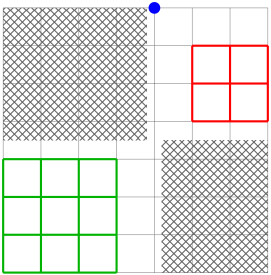
…vaguely

For a class $\mathcal{C}$, the (upper) growth rate is

$$\overline{\mathrm{gr}}(\mathcal{C}) = \limsup_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}.$$

Must exist due to Marcus & Tardos (2004).
Open question: Can lim sup always be replaced with lim?

Stack sortable permutations, $\mathcal{C} = \text{Av}(231)$



Functional equation: $f_{\mathcal{C}}(z) = 1 + f_{\mathcal{C}}(z) \cdot z \cdot f_{\mathcal{C}}(z)$. Solving gives

$$f_{\mathcal{C}}(z) = \frac{1 - \sqrt{1 - 4z}}{2z} = 1 + z + 2z^2 + 5z^3 + 14z^4 + \cdots$$

the generating function for the Catalan numbers $(1, 1, 2, 5, 14, 42, \ldots)$.

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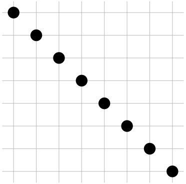
the generating function for the Catalan numbers $(1, 1, 2, 5, 14, 42, \ldots)$.

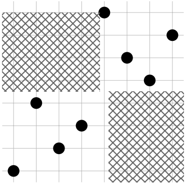
From this, the growth rate is:

$$\begin{aligned} \mathrm{gr}(\mathcal{C}) &= \frac{1}{\sup\{r \geq 0 : f_{\mathcal{C}}(z) \text{ is analytic in } |z| < r\}} \\ &= 4. \end{aligned}$$

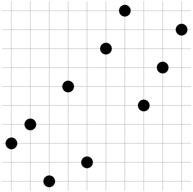
Examples



$$f = \frac{1}{1-z}; \quad \mathrm{gr} = 1.$$



$$f = \frac{1-\sqrt{1-4z}}{2z}; \quad \mathrm{gr} = 4.$$



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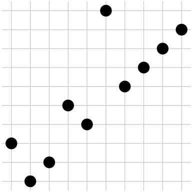

$$f = \frac{1-z}{1-2z}; \quad \mathrm{gr} = 2.$$

Diversion: Principal class growth rates

For a permutation β of length k:

- Stanley & Wilf (1980s): Conjecture there exists c such that

$$|\mathrm{Av}(\beta)_n| \leq c^n.$$

- Arratia (1999): Stanley–Wilf equivalent to existence of $\mathrm{gr}(\mathrm{Av}(\beta))$. Conjectures $c \leq (k-1)^2$.

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- Marcus & Tardos (2004): $c \leq 15^{2k^4\binom{k^2}{k}}$ ($\Rightarrow$ proves Stanley–Wilf).
- Albert, Elder, Rechnitzer, Westcott & Zabrocki (2006): $\mathrm{gr}(\mathrm{Av}(1324)) \geq 9.47$ ($\Rightarrow$ disproves Arratia).

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- Marcus & Tardos (2004): $c \le 15^{2k^4\binom{k^2}{k}}$ ($\Rightarrow$ proves Stanley–Wilf).
- Albert, Elder, Rechnitzer, Westcott & Zabrocki (2006): $\mathrm{gr}(\mathrm{Av}(1324)) \ge 9.47$ ($\Rightarrow$ disproves Arratia).
- Fox (2013+): $c \ge 2^{k^{\theta(1)}}$ for almost all β ($\Rightarrow$ *really* disproves Arratia).

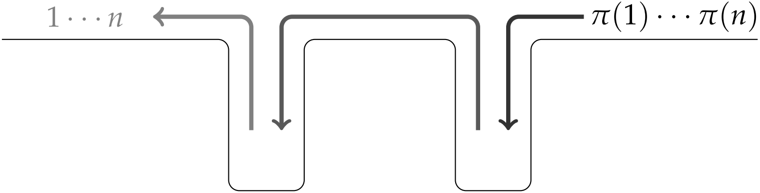
Diversion: Principal class enumeration

β	$f_{\mathrm{Av}(\beta)}(z)$	$\mathrm{gr}(\mathrm{Av}(\beta))$
1	1	0
12	$\frac{1}{1-z}$	1
123 132	$\dfrac{1-\sqrt{1-4z}}{2z}$	4
1342 2413	$\dfrac{1+20z-8z^2+\sqrt{(1-8z)^3}}{2\,(1+z)^3}$	8
1234 1243 1432 2143	$\dfrac{1+5z-\sqrt[4]{(1-9z)^3(1-z)}\,{}_2F_1\left(-\frac{1}{4},\frac{3}{4};1;\frac{64z}{(z-1)(1-9z)^3)}\right)}{6z^2}$	9
1324	?	$\in [10.27, 13.5]$

Up to symmetries, this covers all $\mathrm{Av}(\beta)$ with $|\beta| \leq 4$.

What properties guarantee a class $\mathcal{C}$ has a ‘tame’ enumeration?

Two stacks in series is wild



Murphy (2003) Not finitely based

Albert, Atkinson & Linton (2010) $\mathrm{gr} \in [8.156, 13.374]$.

Pierrot & Rossin (2017) Membership is polynomial time

Elvey Price & Guttman (2017) Exact enumeration to length 20

Estimate: Generating function $\sim A(1 - \mu \cdot z)^{\gamma}$

$\mathrm{gr} \approx 12.5$

§3A D-finite generating functions

Generating functions expressible as solutions to systems of linear homogeneous differential equations with polynomial coefficients.

[Barely tame: Should be kept behind big fences in zoos.]

Noonan–Zeilberger

Recall: $\mathcal{C} = \mathrm{Av}(B)$ is *finitely based* if B is finite.

Conjecture (Noonan & Zeilberger, 1996)

Every finitely based class has a D-finite generating function.

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Conjecture (Noonan & Zeilberger, 1996)

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FALSE

Conjecture (Zeilberger, 2005)

Noonan–Zeilberger is false.

Theorem (Garrabrant, Pak, 2015+)

Zeilberger is right: Noonan–Zeilberger is false.

More wild candidates

Conjecture (Albert, Homberger, Pantone, Shar, Vatter, 2018)

None of the classes $Av(4123, 4231)$, $Av(4123, 4312)$ *or* $Av(4231, 4321)$ *has a D-finite generating function.*

Conjecture is based on analysis of the first 600+ terms of the enumeration sequences.

§3B Algebraic generating functions

Generating functions expressible as solutions to systems of algebraic equations with polynomial coefficients.

[Non-dangerous, but not suitable as pets.]

"...the standard intuition of what a family with an algebraic generating function looks like: the algebraicity suggests that it may (or should...), be possible to give a recursive description of the objects based on disjoint union of sets and concatentation of objects."

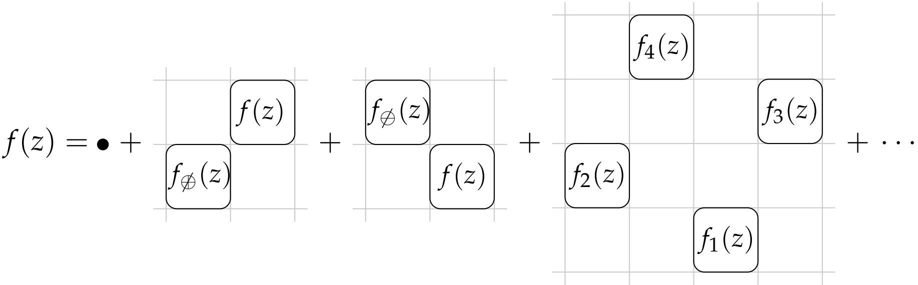
— Bousquet-Mélou, 2006

Canonical example of algebraicity

Theorem (Albert & Atkinson, 2005)

If a class $\mathcal{C}$ contains only finitely many 'simple' permutations, then it has an algebraic generating function and is finitely based.

Think of this as a generalisation of our enumeration of Av(231):



A permutation class is well-quasi-ordered (WQO) if it contains no infinite antichains.

Thought of as a strong indicator of ‘tameness’, so:

A permutation class is well-quasi-ordered (WQO) if it contains no infinite antichains.

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Conjecture (Vatter, 2015)

Every WQO permutation class has an algebraic generating function.

A permutation class is well-quasi-ordered (WQO) if it contains no infinite antichains.

Thought of as a strong indicator of 'tameness', so:

Conjecture (Vatter, 2015)

Every WQO permutation class has an algebraic generating function.

FALSE

Theorem (B. & Vatter, 2025+)

There are uncountably many WQO permutation classes with distinct enumerations.
Hence there exist WQO permutation classes that do not have algebraic (or even D-finite) generating functions.

§3C Rational generating functions

Generating functions of the form $p(z)/q(z)$ with $p, q \in \mathbb{Z}[z]$.

[You can keep these as pets.]

Some pets are *very* dangerous

Theorem (Albert, B., Vatter, 2013)

Every proper permutation class $\mathcal{C}$ is contained in a permutation class with a rational generating function.

Pets are *rare*

Theorem (Bóna, 2020)

Most classes of the form $Av(\beta)$ do not have a rational generating function.

Small(ish) classes

Theorem (Albert, Ruškuc & Vatter, 2015)

Every permutation class with growth rate < 2.20557 *has a rational generating function.*

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Theorem (Albert and Atkinson, 2005)

Every proper subclass of $Av(231)$ *is finitely based and has a rational generating function.*

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Every permutation class with growth rate < 2.20557 *has a rational generating function.*

Theorem (Albert and Atkinson, 2005)

Every proper subclass of $Av(231)$ *is finitely based and has a rational generating function.*

Theorem (Albert, B., Ruškuc & Vatter, 2019)

Every proper subclass of $Av(321)$ *that is finitely based has a rational generating function.*

Recall: Both $\mathrm{Av}(231)$ and $\mathrm{Av}(321)$ have growth rate 4…

Conjecture (B.)

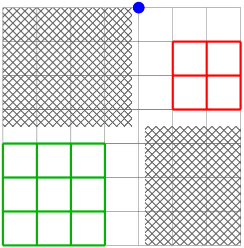
Every finitely based permutation class with growth rate < 4 *has a rational generating function.*

Conjecture (B.)

Every finitely based permutation class with growth rate < 4 has a rational generating function.

- False if we drop ‘finitely based’ requirement. (Fails at gr $= 2.20557$.)
- B. & Opler (ongoing): True for growth rate < 2.61803

Functional equations for algebraic $\mathcal{C} = \mathrm{Av}(231)$



Functional equation: $f(z) = 1 + f(z) \cdot z \cdot f(z)$ has (nonrational) solution

$$f(z) = \frac{1 - \sqrt{1-4z}}{2z} = 1 + z + 2z^2 + 5z^3 + 14z^4 + \cdots .$$

In classes with growth rate less than 4, we can't get functional equations involving $f^2(z)$ terms.

So, *if* we can get a functional equation for some class with $\mathrm{gr} < 4$, then we likely get a rational generating function.

Thanks!

References 1

M. H. Albert and M. D. Atkinson. Simple permutations and pattern restricted permutations.
Discrete Math., 300(1-3):1–15, 2005

M. H. Albert, M. Atkinson, and S. Linton. Permutations generated by stacks and deques.
Ann. Comb., 14(1):3–16, 2010

M. Albert, R. Brignall, N. Ruškuc, and V. Vatter. Rationality for subclasses of 321-avoiding permutations.
European Journal of Combinatorics, 78:44–72, 2019

M. H. Albert, R. Brignall, and V. Vatter. Large infinite antichains of permutations.
Pure Math. Appl. (PU.M.A.), 24(2):47–57, 2013

M. H. Albert, M. Elder, A. Rechnitzer, P. Westcott, and M. Zabrocki. On the Wilf-Stanley limit of 4231-avoiding permutations and a conjecture of Arratia.
Adv. in Appl. Math., 36(2):95–105, 2006

M. H. Albert, C. Homberger, J. Pantone, N. Shar, and V. Vatter. Generating permutations with restricted containers.
J. Combin. Theory Ser. A, 157:205–232, 2018

References 2

M. H. Albert, N. Ruškuc, and V. Vatter. Inflations of geometric grid classes of permutations.
Israel J. Math., 205(1):73–108, 2015

R. Arratia. On the Stanley-Wilf conjecture for the number of permutations avoiding a given pattern.
Electron. J. Combin., 6:Note 1, 4 pp., 1999

M. Bóna. Supercritical sequences, and the nonrationality of most principal permutation classes.
European J. Combin., 83:103020, 8, 2020

R. Brignall and V. Vatter. Uncountably many enumerations of well-quasi-ordered permutation classes.
Arxiv:2211.12397, 2023

S. Corteel and L. K. Williams. Tableaux combinatorics for the asymmetric exclusion process.
Adv. in Appl. Math., 39(3):293–310, 2007

A. Elvey Price and A. J. Guttmann. Permutations sortable by two stacks in series.
Adv. in Appl. Math., 83:81–96, 2017

J. Fox. Stanley-wilf limits are typically exponential.
ArXiv:1310.8378, 2013

References 3

S. Garrabrant and I. Pak. Pattern avoidance is not P-recursive.
ArXiv:1505.06508

V. Lakshmibai and B. Sandhya. Criterion for smoothness of Schubert varieties in $\mathrm{SL}(n)/B$.
Proc. Indian Acad. Sci. Math. Sci., 100(1):45–52, 1990

A. Marcus and G. Tardos. Excluded permutation matrices and the Stanley-Wilf conjecture.
J. Combin. Theory Ser. A, 107(1):153–160, 2004

M. M. Murphy. *Restricted permutations, antichains, atomic classes, and stack sorting*.
PhD thesis, Univ. of St Andrews, 2002

J. Noonan and D. Zeilberger. The enumeration of permutations with a prescribed number of “forbidden” patterns.
Adv. in Appl. Math., 17(4):381–407, 1996

A. Pierrot and D. Rossin. 2-stack sorting is polynomial.
Theory Comput. Syst., 60(3):552–579, 2017

V. Vatter. Permutation classes.
In *Handbook of enumerative combinatorics*, Discrete Math. Appl. (Boca Raton), pages 753–833. CRC Press, Boca Raton, FL, 2015