Infinite Antichains and Partial Well-Order in Permutation Classes

Robert Brignall

Department of Mathematics
University of Bristol

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Hereditary Properties

- Graphs: containment as an induced subgraph gives a quasi-order.
- Hereditary properties: downsets in this order. e.g. K_4-free graphs.

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- Hereditary properties: downsets in this order. e.g. K_4-free graphs.

- Permutations: pattern containment ordering is a partial order.
- Permutation classes: downsets in this order. e.g. 231-avoiding permutations.

Pattern Containment

- A permutation $\tau = \tau(1) \cdots \tau(k)$ is contained in the permutation $\sigma = \sigma(1)\sigma(2) \cdots \sigma(n)$ if there exists a subsequence $\sigma(i_1)\sigma(i_2) \cdots \sigma(i_k)$ order isomorphic to τ.

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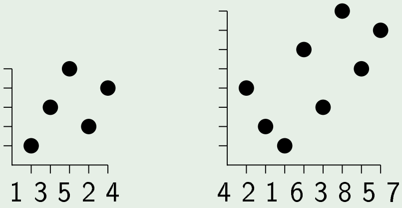

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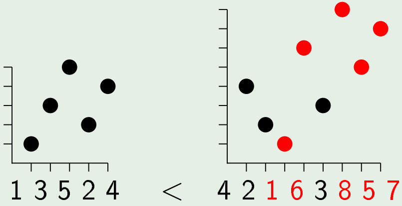

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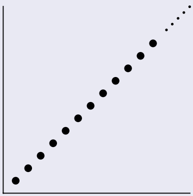
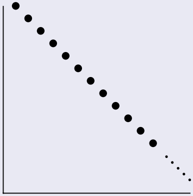
- Downsets of permutations in this partial order form permutation classes.
  i.e. $\pi \in \mathcal{C}$ and $\sigma \leq \pi$ implies $\sigma \in \mathcal{C}$.
- Typical description: basis is the set of minimal excluded elements.
  $\mathcal{C} = \mathrm{Av}(B) = \{\pi : \beta \not\leq \pi \text{ for all } \beta \in B\}$.

Cherry-picked Examples

- $\mathrm{Av}(21) = \{1, 12, 123, 1234, \ldots\}$, the increasing permutations.
- $\mathrm{Av}(12) = \{1, 21, 321, 4321, \ldots\}$, the decreasing permutations.

Typical Elements

Cherry-picked Examples

- $\oplus 21 = \text{Av}(321, 312, 231) = \{1, 12, 21, 123, 132, 213, \ldots\}.$
- $\ominus 12 = \text{Av}(123, 213, 132) = \{1, 12, 21, 231, 312, 321, \ldots\}.$

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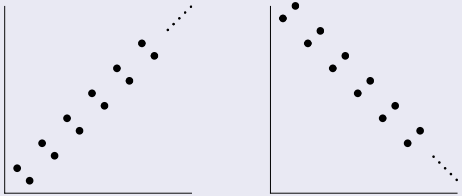

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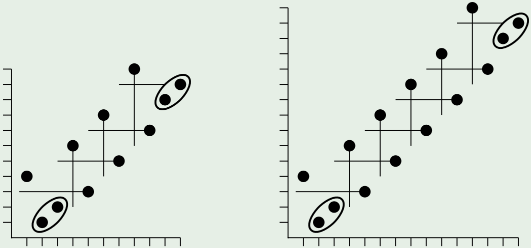
Example (Increasing Oscillating Antichain)



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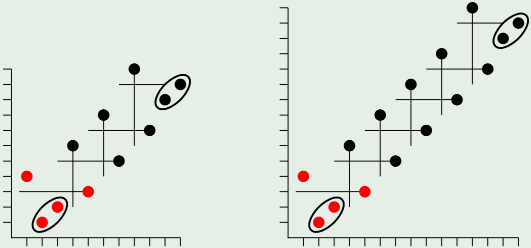

- Bottom copies of 4123 must match up (the anchor).

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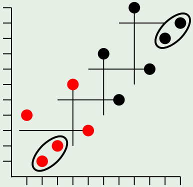

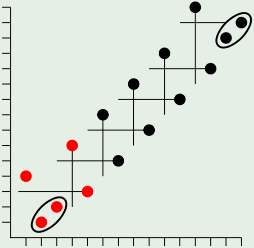

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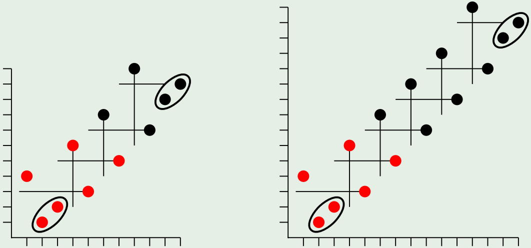

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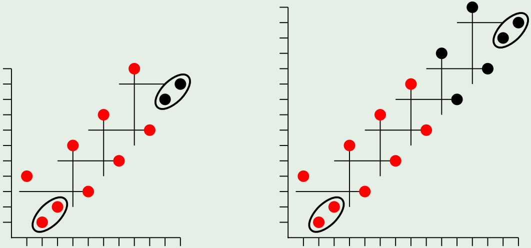

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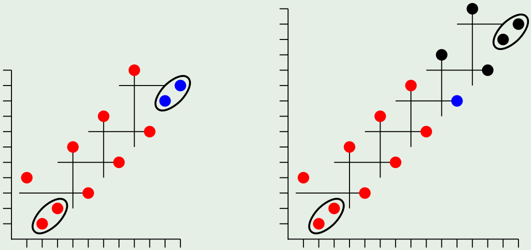

- Last pair cannot be embedded.

When are there antichains?

No infinite antichains.

- Words over a finite alphabet [Higman].
- Graphs closed under minors [Robertson and Seymour].

Infinite antichains.

- Graphs closed under induced subgraphs (or merely subgraphs). e.g. $C_3, C_4, C_5, \ldots$
- Permutations closed under containment.
- Tournaments, digraphs, ...

Partial Well Order

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Question

Can we decide whether a permutation class given by a finite basis is pwo?

- To prove pwo — Higman's theorem is useful.
- To prove not pwo — find an antichain.

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Can we decide whether a hereditary property given by a finite basis is wqo?

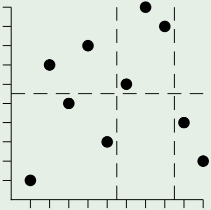
- To prove pwo — Higman's theorem is useful.
- To prove not pwo — find an antichain.
- Other structures: well quasi-order, not pwo, but same idea.

Grid Classes

- Matrix $\mathcal{M}$ whose entries are permutation classes.
- $\mathrm{Grid}(\mathcal{M})$ the grid class of $\mathcal{M}$: all permutations which can be "gridded" so each cell satisfies constraints of $\mathcal{M}$.

Example

- Let $\mathcal{M} = \begin{pmatrix} \mathrm{Av}(21) & \mathrm{Av}(231) & \emptyset \\ \mathrm{Av}(123) & \emptyset & \mathrm{Av}(12) \end{pmatrix}$.



$\in \mathrm{Grid}(\mathcal{M})$

Grid classes are useful

- $\mathcal{C}_n =$ permutations in the class $\mathcal{C}$ of length n.
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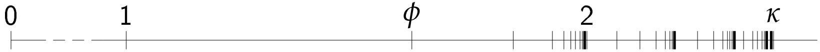

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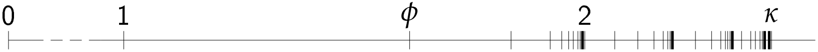
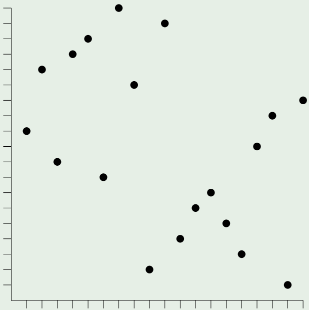


- Cf "canonical properties" of graphs [Balogh, Bollobás and Weinreich].

- Special case: all cells of $\mathcal{M}$ are $\mathrm{Av}(21)$ or $\mathrm{Av}(12)$.
- Rewrite $\mathcal{M}$ as a matrix with entries in $\{0, 1, -1\}$.

Example

$$\mathcal{M} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$



Monotone Grid Classes

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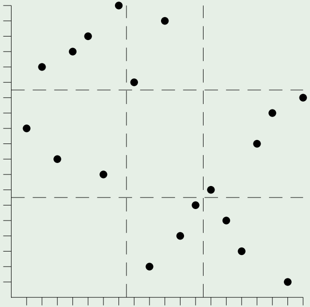

The Graph of a Matrix

- Graph of a matrix, $G(\mathcal{M})$, formed by connecting together all non-zero entries that share a row or column and are not "separated" by any other nonzero entry.

Example

$$\begin{pmatrix} \mathcal{C} & 0 & 0 & \mathcal{D} \\ 0 & 0 & \mathcal{E} & 0 \\ \mathcal{D} & \mathcal{E} & 0 & \mathcal{C} \\ 0 & 0 & 0 & \mathcal{D} \end{pmatrix}$$

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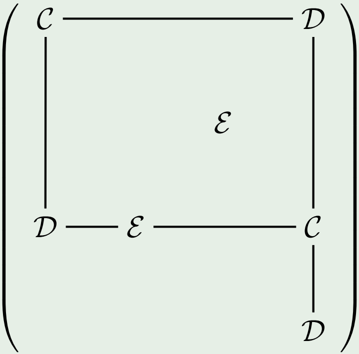
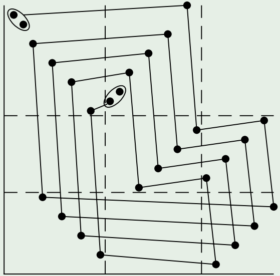


Monotone Grids and Partial Well Order

Theorem (Murphy and Vatter, 2003)

The monotone grid class $\mathrm{Grid}(\mathcal{M})$ *is pwo if and only if* $G(\mathcal{M})$ *is a forest, i.e.* $G(\mathcal{M})$ *contains no cycles.*

Antichain Construction



When does that apply?

Question

When is a class $\mathcal{C}$ (a subset of) a monotone grid class?

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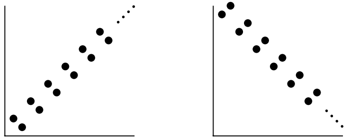
Answer [Vatter]

A class $\mathcal{C}$ is monotone griddable if and only if it contains neither the classes $\oplus 21$ nor $\ominus 12$.

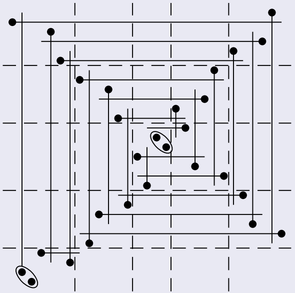


Two is too many

Theorem

A grid class whose graph has a component containing two or more non-monotone-griddable classes is not pwo.

Proof.



- Antichain element.

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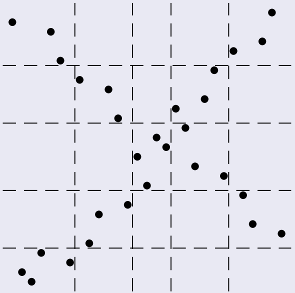

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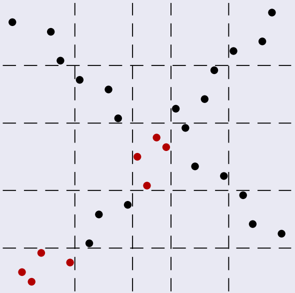

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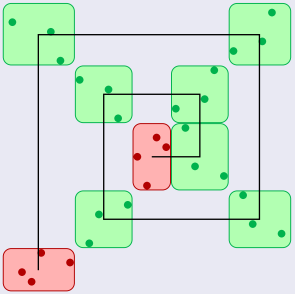

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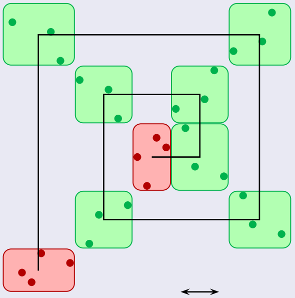

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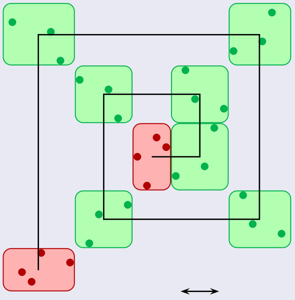

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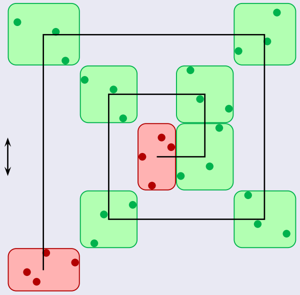

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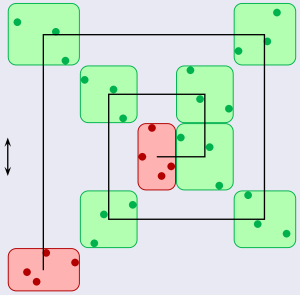

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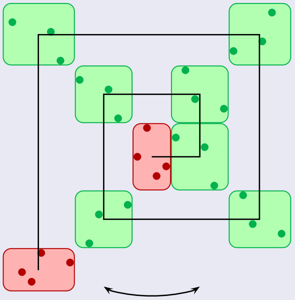

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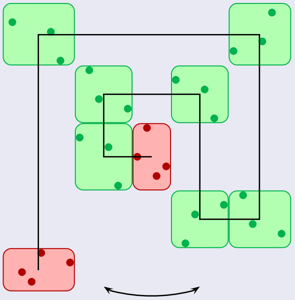

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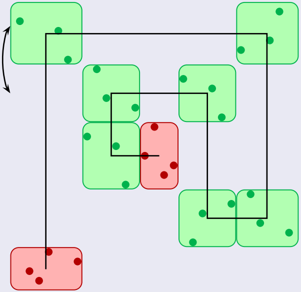

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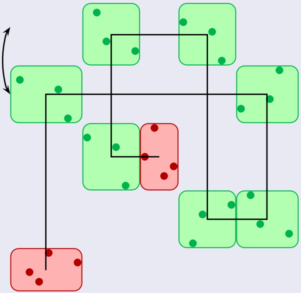

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Just one non-monotone

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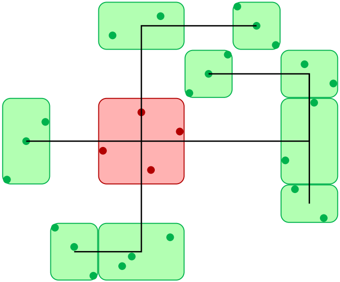

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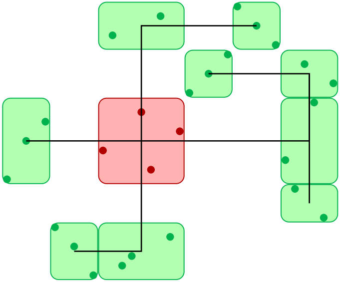
- Form a rooted tree on the red cell, and use Higman's Theorem.



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Question

Can we decide whether a permutation class given by a finite basis is pwo?

- We're closer to answering this, but still some way off.
- Try doing this for your favourite combinatorial structure.

Thanks!