Pancakes and superpermutations

An invitation to permutation patterns

Robert Brignall

M500 Revision Weekend, Kent's Hill

11th May 2024

The Open University

The pancake sorting problem

The chef in our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes. Therefore, when I deliver them to a customer, on the way to the table I rearrange them (so that the smallest winds up on top, and so on, down to the largest at the bottom) by grabbing several from the top and flipping them over, repeating this (varying the number I flip) as many times as necessary.

If there are n pancakes, what is the maximum number of flips (in terms of n) that I will ever have to use to rearrange them?

Jacob E. Goodman (a.k.a. Harry Dweighter), 1975

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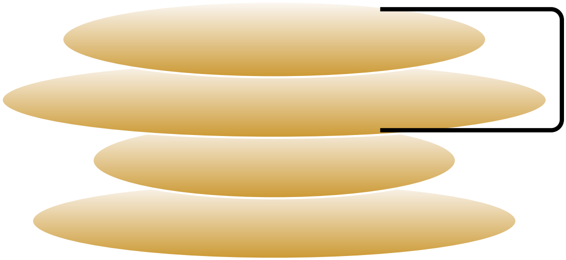

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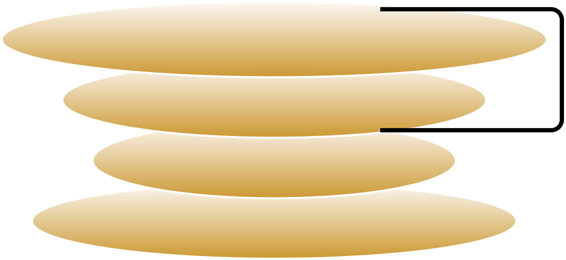

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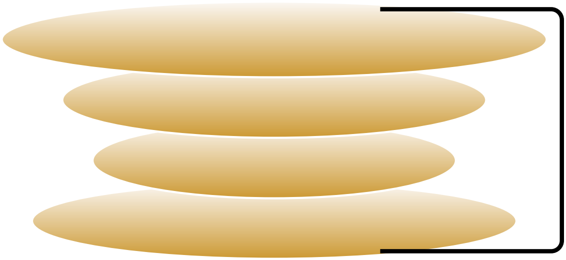

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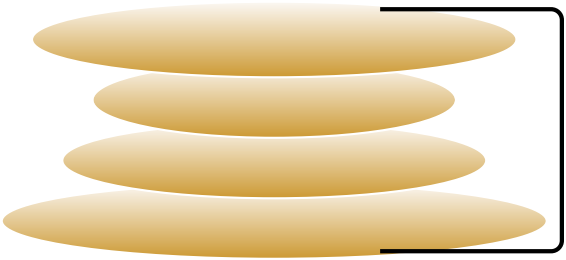

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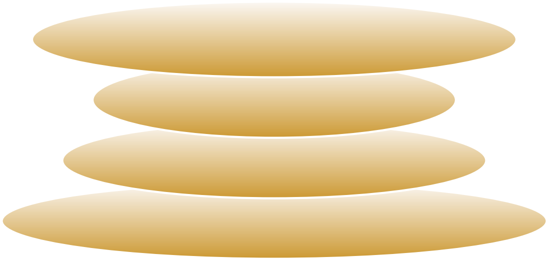

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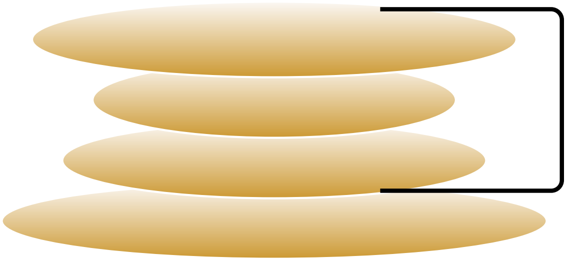

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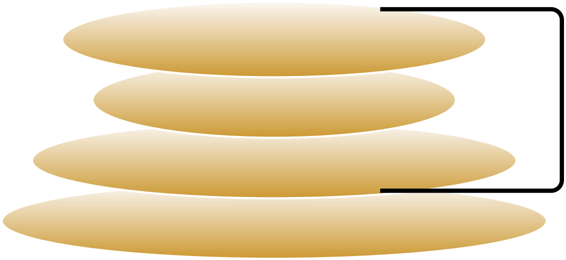

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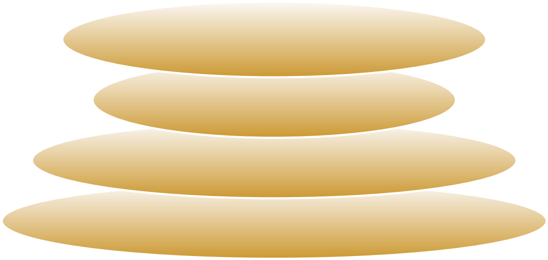

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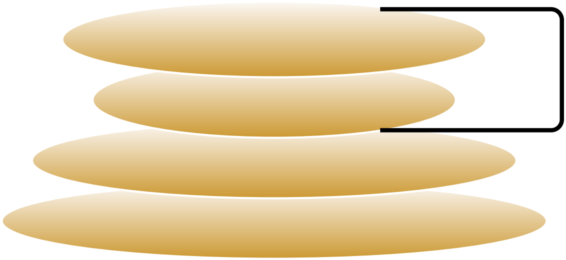

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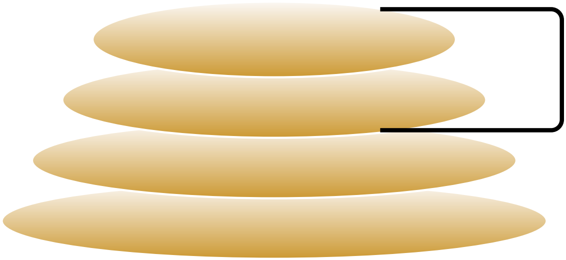

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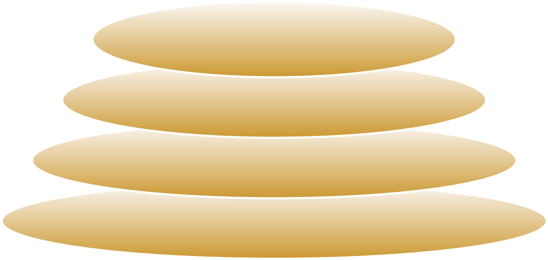

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Pancakes and permutations

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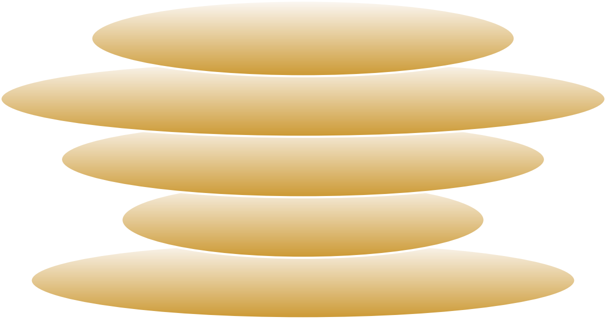

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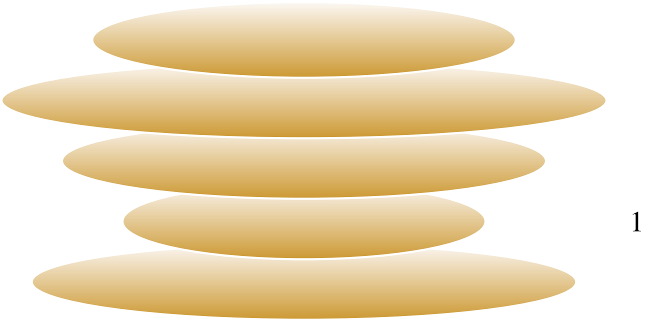

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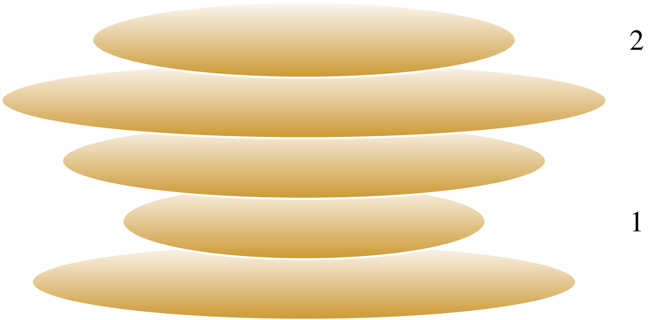

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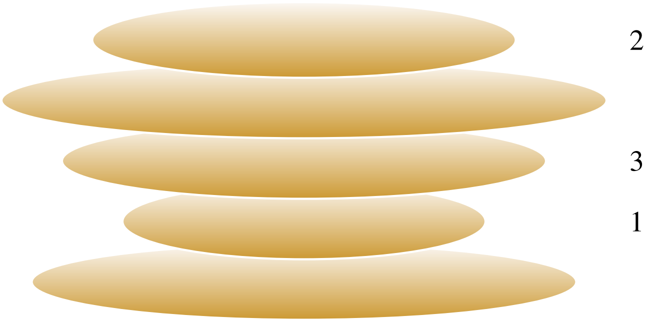

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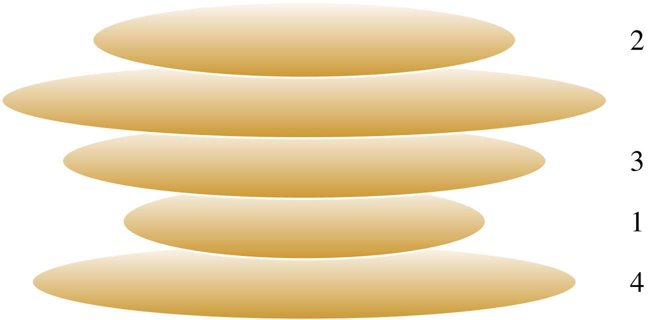

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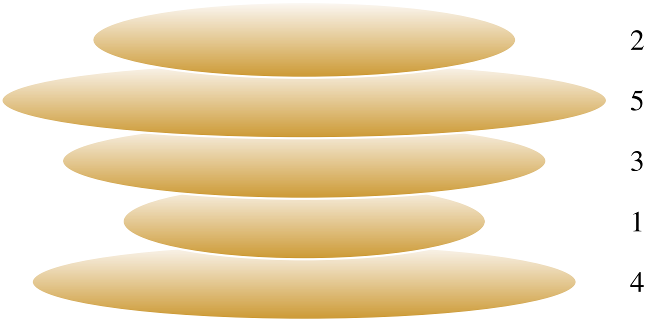

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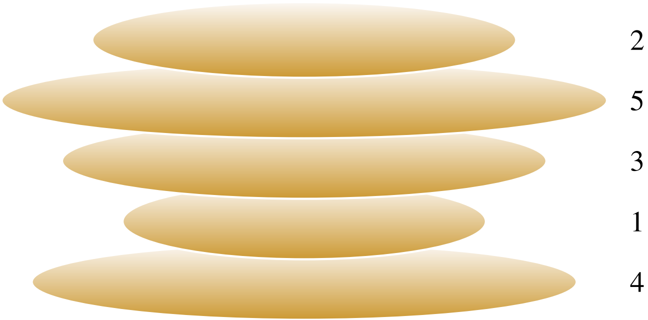

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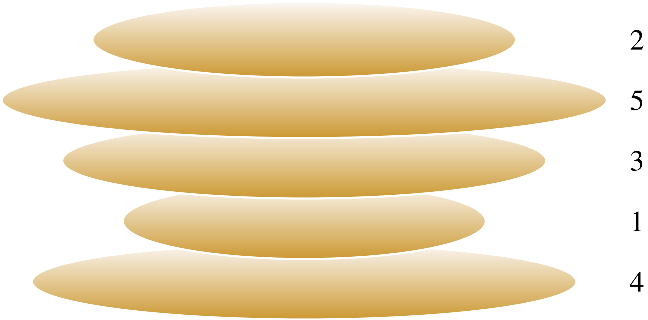

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A sorted pancake stack would be 1 2 3 4 5.

Obligatory maths slide

For us, a permutation of length n is the symbols $1, 2, \ldots, n$ in some order.

Example: $\pi = 3\,1\,4\,5\,9\,2\,6\,8\,7$

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There are $n! = n(n-1)(n-2)\cdots 1$ permutations of length n.

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123, 132, 213, 231, 312, 321

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The pancake flip operation is a prefix reversal:

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Let $f(\pi)$ denote the number of flips needed to turn a permutation (or pancake) π into $12\cdots n$.

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Let f_n denote the worst case for length n. That is,

$$f_n = \max_{\substack{\pi \text{ of} \\ \text{length } n}} f(\pi).$$

Small n:

$f_1 = 0$ (no flips needed!)
$f_2 = 1$ $(21 \to 12)$
$f_3 = 3$ $(132 \to 312 \to 213 \to 123)$
$f_4 = 4$ $(3142 \to 4132 \to 2314 \to 3214 \to 1234)$

Bounding f_n

We want:

$$L_n \quad \leqslant \quad f_n \quad \leqslant \quad U_n$$

where

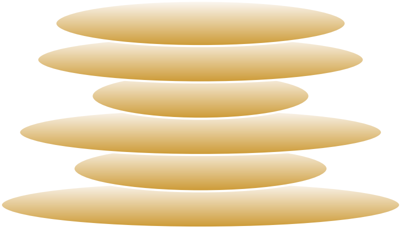
U_n is an upper bound: need algorithm to sort *any* length n permutation in $\leqslant U_n$ flips.

L_n is a lower bound: need a length n permutation requiring L_n flips.

Upper bound: A simple algorithm

1. Find the *biggest* pancake that's in the wrong place.
2. Flip this biggest pancake to the top.
3. Now flip it into the correct position.

Example:



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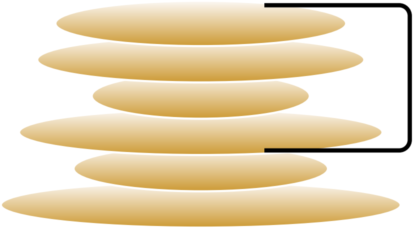

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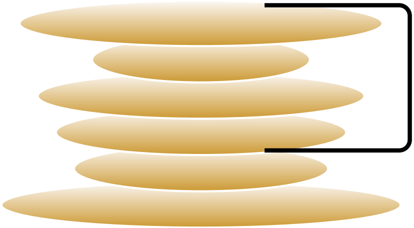

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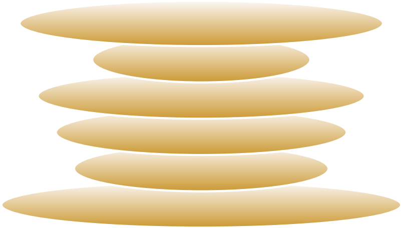

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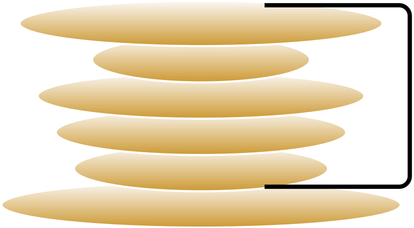

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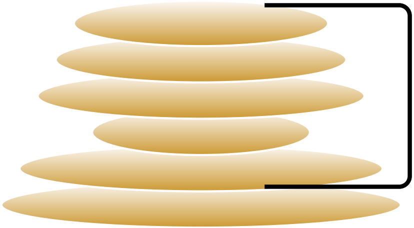

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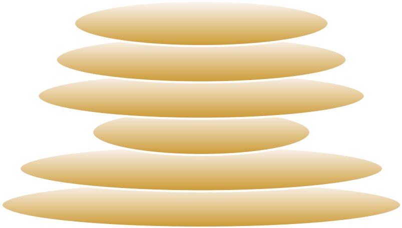

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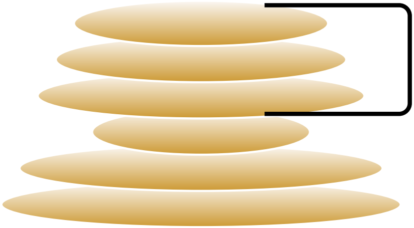

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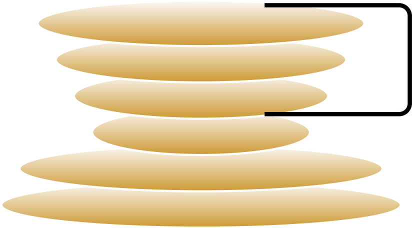

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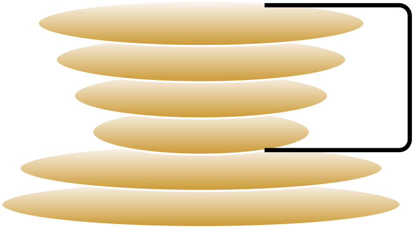

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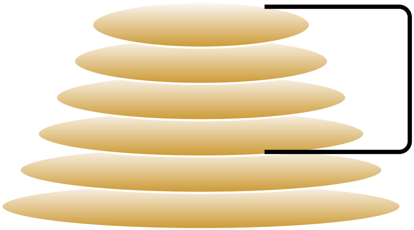

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Example:

Worst case: Every pancake is in the wrong position. 2 flips to fix, so

$$U_n \leqslant 2n - 3.$$

A better upper bound

BOUNDS FOR SORTING BY PREFIX REVERSAL

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Received 18 January 1978
Revised 28 August 1978

For a permutation σ of the integers from 1 to n, let $f(\sigma)$ be the smallest number of prefix reversals that will transform σ to the identity permutation, and let $f(n)$ be the largest such $f(\sigma)$ for all σ in (the symmetric group) S_n. We show that $f(n) \leqslant (5n+5)/3$, and that $f(n) \geqslant 17n/16$ for n a multiple of 16. If, furthermore, each integer is required to participate in an even number of reversed prefixes, the corresponding function $g(n)$ is shown to obey $3n/2-1 \leqslant g(n) \leqslant 2n+3$.

$$\text{So } U_n \leqslant \frac{5n+5}{3} \approx 1.6667n.$$

A better better upper bound

An (18/11)n upper bound for sorting by prefix reversals

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A R T I C L E I N F O

Keywords:
Pancake problem
Pancake network
Sorting by prefix reversals
Permutations
Upper bounds

A B S T R A C T

The pancake problem asks for the minimum number of prefix reversals sufficient for sorting any permutation of length n. We improve the upper bound for the pancake problem to $(18/11)n + O(1) \approx (1.6363)n$.

$$\text{So } U_n \leqslant \frac{18n}{11} + \text{a bit} \approx 1.6363n.$$

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Each flip increases #adjacencies by $\leqslant 1$.

So any permutation with zero adjacencies will need at least $n-1$ flips.

$$L_n \geqslant n-1.$$

Example: $\pi = 2\,4\,6\ \cdots n\,1\,3\,5\ \cdots\ n-1$ $\quad$ (n even)

A better lower bound

Similar (but more complicated) ideas gives:

$$L_n \geqslant \frac{15n}{14} \approx 1.0714n$$

for $n \geqslant 6$.

State-of-the-art

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For small n (by computer search):

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
f_n	0	1	3	4	5	7	8	9	10	11	13	14	15	16	17	18	19	20	22

Applications: genomics

Transforming Cabbage into Turnip: Polynomial Algorithm for Sorting Signed Permutations by Reversals

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Bioinformatics, SmithKline Beecham Pharmaceuticals, King of Prussia, Pennsylvania

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University of Southern California, Los Angeles, California

Abstract. Genomes frequently evolve by reversals $\rho(i,j)$ that transform a gene order $\pi_1 \cdots \pi_i \pi_{i+1} \cdots \pi_{j-1} \pi_j \cdots \pi_n$ into $\pi_1 \cdots \pi_i \pi_{j-1} \cdots \pi_{i+1} \pi_j \cdots \pi_n$. Reversal distance between permutations π and σ is the minimum number of reversals to transform π into σ. Analysis of genome rearrangements in molecular biology started in the late 1930's, when Dobzhansky and Sturtevant published a milestone paper presenting a rearrangement scenario with 17 inversions between the species of *Drosophila*. Analysis of genomes evolving by inversions leads to a combinatorial problem of *sorting by reversals* studied in detail recently. We study sorting of *signed* permutations by reversals, a problem that adequately models rearrangements in small genomes like chloroplast or mitochondrial DNA. The previously suggested approximation algorithms for sorting signed permutations by reversals compute the *reversal distance* between permutations with an astonishing accuracy for both simulated and biological data. We prove a duality theorem explaining this intriguing performance and show that there exists a "hidden" parameter that allows one to compute the reversal distance between signed permutations in polynomial time.

ELSEVIER

Discrete Applied Mathematics 61 (1995) 105–120

DISCRETE APPLIED MATHEMATICS

On the problem of sorting burnt pancakes

David S. Cohen*,[1], Manuel Blum[2]

Computer Science Division, University of California, Berkeley, CA 94720, USA

Received 30 June 1992; revised 5 October 1993

Abstract

number of flips required in the worst case to sort a stack of n pancakes. Equivalently, we seek bounds on the number of *prefix reversals* necessary to sort a list of n elements. Bounds of $17n/16$ and $(5n+5)/3$ were shown by Gates and Papadimitriou in 1979. In this paper, we consider a traditional variation of the problem in which the pancakes are two sided (one side is "burnt"), and must be sorted to the size-ordered configuration in which every pancake has its burnt side down. Let $g(n)$ be the number of flips required to sort n "burnt pancakes". We find that $3n/2 \leqslant g(n) \leqslant 2n-2$, where the upper bound holds for $n \geqslant 10$. We consider the conjecture that the most difficult case for sorting n burnt pancakes is $-I_n$, the configuration having the pancakes in proper size order, but in which each individual pancake is upside down. We present an algorithm for sorting $-I_n$ in $23n/14+c$ flips, where c is a small constant, thereby establishing a bound of $g(n) \leqslant 23n/14+c$ under the conjecture. Furthermore, the longstanding upper bound of $f(n)$ is also improved to $23n/14+c$ under the conjecture.

Superpermutations

Meet Haruhi Suzumiya

The Haruhi problem

What is the least number of Haruhi episodes that you would have to watch in order to see the original 14 episodes in every order possible?

The *generalised* Haruhi problem

What is the least number of Haruhi episodes that you would have to watch in order to see n distinct episodes in every order possible?

The Haruhi problem

What is the least number of Haruhi episodes that you would have to watch in order to see the original 14 episodes in every order possible?

In maths terms, we want to find the shortest superpermutation for each n.

Example: $n = 3$

Watch episodes 1, 2 and 3 in these orders:

$$123 \quad 132 \quad 213 \quad 231 \quad 312 \quad 321$$

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The following sequence of length 9 contains all 6 of these:

$$1 \quad 2 \quad 3 \quad 1 \quad 2 \quad 1 \quad 3 \quad 2 \quad 1$$

Example: $n = 3$

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The following sequence of length 33 contains all 24 of these:

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Shortest superpermutations

For $n = 1, 2, 3, 4, 5$, the shortest superpermutations have lengths

$$1,\ 3,\ 9,\ 33,\ 153.$$

For $n = 6$, the shortest *known* has length 872. Best lower bound is 867.

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For $n = 7$, the shortest known has length 5906. Best lower bound is 5884.

For $n = 14$, it's between 93 884 313 611 and 93 924 230 411.

Upper bounds (via explicit construction)

Best upper bound ($n \geqslant 7$):

$$n! + (n-1)! + (n-2)! + (n-3)! + n - 3.$$

gregegan.net/SCIENCE/Superpermutations/Superpermutations.html

Superpermutations

by Greg Egan

```
ERMUTATUIOSNPSERP
RMUTATUIOSNPSERPE
MUTATUIOSNPSERPER
UTATUIOSNPSERPERM
TATUIOSNPSERPERMU
ATUIOSNPSERPERMUT
TUIOSNPSERPERMUTA
```

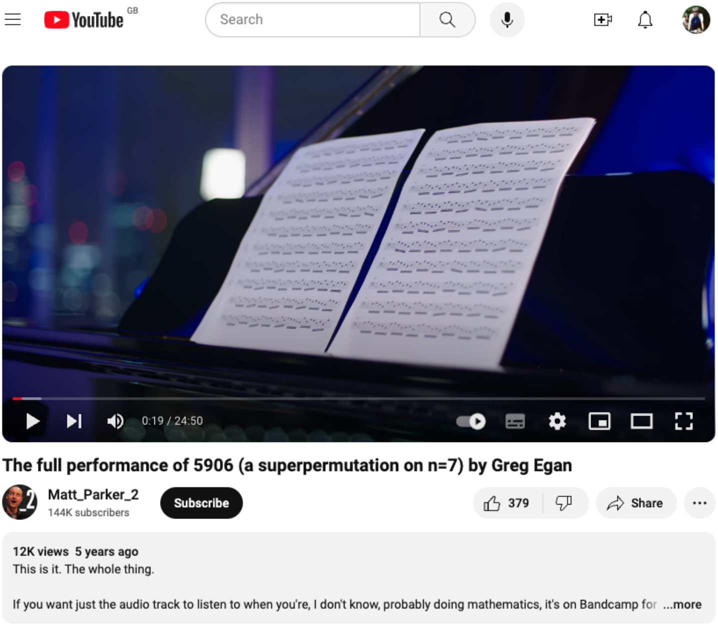
Very soon after Chaffin's result, Robin Houston announced[4] the discovery of a superpermutation for n=6 with only 872 characters, one less than $L(6)$=873. He found this by treating the construction of superpermutations as an example of the Travelling Salesman Problem, and using algorithms designed to generate solutions to that problem. So the original minimal superpermutation conjecture was proved false!

By applying the usual recursion to Houston's shorter n=6 superpermutations, it becomes possible to generate superpermutations for any greater value of n that are also one character shorter than $L(n)$.

However, it turns out that for n≥7, there is a way to do even better. By adapting a construction devised by Aaron Williams[5] in 2013, it's possible to generate superpermutations of length:

$$L_2(n) = n! + (n-1)! + (n-2)! + (n-3)! + n - 3$$

$n = 7$ record published by … piano?



Lower bounds

Easy lower bound:

$$n! + (n-1)$$

Proof: There are $n!$ permutations of length n. Each must start at a different position in the superpermutation, so that's $n!$ symbols.
After the last permutation starts, there must be $n-1$ more symbols to complete this final permutation. □

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After the last permutation starts, there must be $n-1$ more symbols to complete this final permutation. $\square$

Now things get a little weird...

On an anime fan website far far away…

>> **Lower bounds** Anonymous Sat, Sep 17, 2011 03:35:54 No.3751197

Quoted by: >>3751366 >>3751370 >>3752047_2 >>3752047_25 >>3752047_43 >>3752047_64 >>3752047_69 >>3752047_74 >>3752047_78 >>3752047_1351 >>3752047_1362 >>3752047_1503 >>3752047_3067 >>3752047_3804 >>3752047_3827

I think I have a proof of the lower bound n! + (n-1)! + (n-2)! + n-3 (for <span class="math">n \geq 2[/spoiler]). I'll need to do this in multiple posts. Please look it over for any loopholes I might have missed.

As in other posts, let n (lowercase) = the number of symbols; there are n! permutations to iterate through.

The obvious lower bound is n! + n-1. We can obtain this as follows:

Let
L = the running length of the string
<span class="math">N_0[/spoiler] = the number of permutations visited
<span class="math">X_0 = L - N_0[/spoiler]

When you write down the first permutation, <span class="math">X_0[/spoiler] is already n-1. For each new permutation you visit, the length of the string must increase by at least 1. So <span class="math">X_0[/spoiler] can never decrease. At the end, <span class="math">N_0 = n![/spoiler], giving us <span class="math">L \geq n! + n-1[/spoiler].

I'll use similar methods to go further, but first I'll need to explain my terminology...

A lower bound on the length of the shortest superpattern

Anonymous 4chan Poster, Robin Houston, Jay Pantone, and Vince Vatter

October 25, 2018

This proof is inspired by that posted anonymously at

http://mathsci.wikia.com/wiki/The_Haruhi_Problem

which itself was taken from a 4chan discussion archived at

https://warosu.org/sci/thread/S3751105#p3751197

Theorem (anonymous)

Every superpermutation for the permutations of length n has length at least

$$n! + (n-1)! + (n-2)! + n - 3.$$

superpermutation - Google S

google.co.uk/search?sca_esv=25cd74cf61df00be...

Google

superpermutation

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WIRED

How a 4chan Post Helped Solve a 25-Year-Old Math Puzzle

In September 16, 2011, an anime fan posted a math question to the online bulletin board 4chan about the cult classic television series The...

11 Nov 2018

The Verge

An anonymous 4chan post could help solve a 25-year-old math mystery

A 4chan poster might have solved part of a very tricky math problem that mathematicians have been working on for at least 25 years.

24 Oct 2018

Indy100

Anonymous anime fan might just have solved a question that's stumped maths experts for 25 years

Somehow, someway, a fan of an obscure Japanese anime series has managed to inadvertently solve a complex maths question which has stumped...

31 Oct 2018

GIGAZINE

Thanks to ``The Melancholy of Haruhi Suzumiya", a difficult math problem that has not been solved for 25 years may be ...

Mathematicians around the world have expressed their belief that a discussion on the overseas bulletin board 4chan may solve a difficult...

Summary: state-of-the-art superpermutations

For $n = 6$ the shortest superpermutation has length between 867 and 872.
(*Aside*: Over 100M CPU hours did not yield a better construction.)

For $n \geqslant 7$, the shortest superpermutation has length between

$$n! + (n-1)! + (n-2)! + n - 3$$

and

$$n! + (n-1)! + (n-2)! + (n-3)! + n - 3.$$

Thanks!