Antichains of Permutations

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Pattern Containment

- A permutation $\tau = t_1 t_2 \dots t_k$ is contained in the permutation $\sigma = s_1 s_2 \dots s_n$ if there exists a subsequence $s_{i_1}, s_{i_2}, \dots, s_{i_k}$ order isomorphic to τ.

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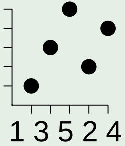
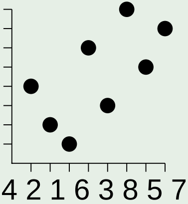
Example





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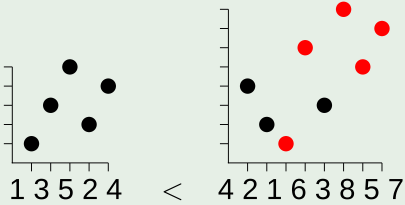

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Example

The class $\mathcal{C} = \mathrm{Av}(12)$ consists of all the decreasing permutations:

$$\{1, 21, 321, 4321, \ldots\}$$

Antichains

- Set of pairwise incomparable permutations.

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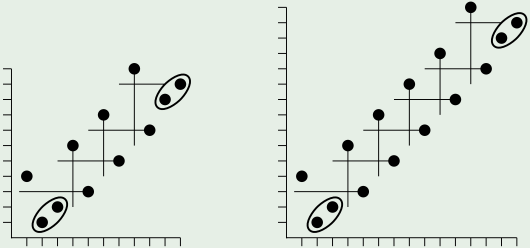
Example (Increasing Oscillating Antichain)



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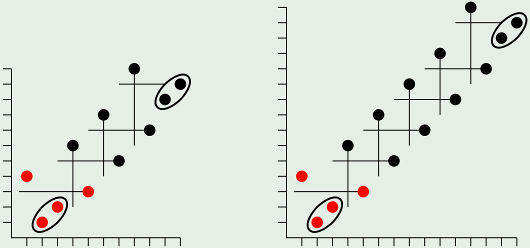

- Bottom copies of 4123 must match up (the anchor).

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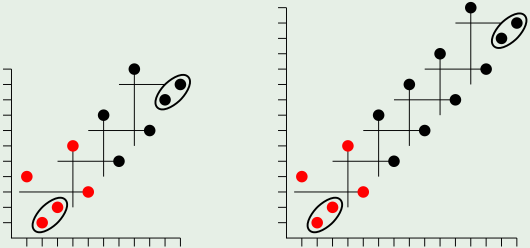

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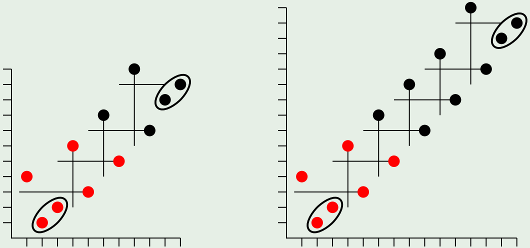

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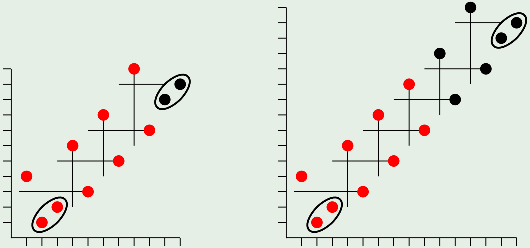

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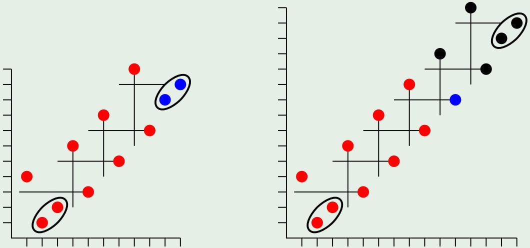

- Last pair cannot be embedded.

Complete and Fundamental Antichains

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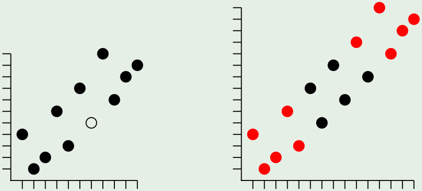
Example

The increasing oscillating antichain is fundamental, but not complete.



Not complete: $I \cup \{321\}$ is an antichain.

More on Fundamental Antichains

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A is fundamental if and only if the proper closure $Cl(A) \setminus A$ is pwo and for every $\pi \in Cl(A) \setminus A$ the set $A^{||\pi}$ is finite.

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- This condition means that terms of a fundamental antichain look "similar".

Conjectures: Antichain Structure

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Conjecture

Every member of a fundamental antichain contains at most two proper intervals.

An Ordering on Antichains

- Define an order on antichains:

$$B \preceq A \Leftrightarrow \text{ for every } \alpha \in A, \text{ there exists } \beta \in B \text{ with } \beta \leq \alpha$$

- Note that $A \subseteq B$ implies $B \preceq A$!
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Lemma

An antichain is minimal under $\preceq$ if and only if it is complete and fundamental.

Partial Well Order

- A permutation class is partially well-ordered (pwo) if it contains no infinite antichains.

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Can we decide whether a permutation class given by a finite basis is pwo?

- To prove pwo — Higman's theorem is useful.
- To prove not pwo — find an antichain.

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Proposition (Nash-Williams, 1963)

Every non-pwo permutation class contains an antichain that is minimal under $\preceq$.

Corollary

Every non-pwo permutation class contains a fundamental antichain.

More on Minimal Antichains

Theorem (Cherlin and Latka, 2000)

For each natural number k, there is a finite set Λ_k of antichains minimal under $\preceq$ with the property that a class avoiding exactly k permutations is pwo if and only if its intersection with each antichain in Λ_k is finite.

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- Caveat: algorithm is not known.

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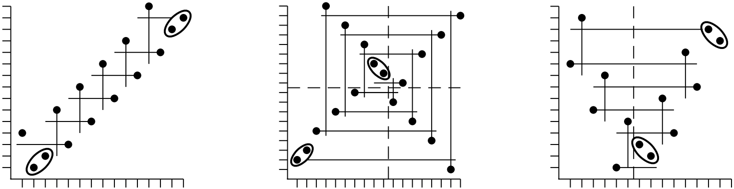
- For permutation classes, Λ_1 consists of the minimal antichains containing increasing oscillating, Widdershins and V.



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Proposition (Atkinson, Murphy and Ruškuc, 2002)

$\mathrm{Av}(\beta)$ *is pwo if and only if* $\beta \in \{1, 12, 21, 132, 213, 231, 312\}$

Topology

- $\mathcal{A}$ — set of all minimal antichains, viewed as a topological space.
- Open sets: for B a finite set of permutations

$$\mathcal{A}_B = \{A \in \mathcal{A} : A \cap \mathrm{Av}(B) \text{ is infinite}\}.$$

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- Equivalence relation:

$$A_1 \rho A_2 \Leftrightarrow \{\mathcal{A}_B : A_1 \in \mathcal{A}_B\} = \{\mathcal{A}_B : A_1 \in \mathcal{A}_B\}.$$

- Easier: $A_1 \rho A_2$ iff $\mathrm{Cl}(A_1) \setminus A = \mathrm{Cl}(A_2) \setminus A$.
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- Quotient: $\mathcal{A}' = \mathcal{A}/\rho$ (is a T_0 space).
- $A \in \mathcal{A}$ is isolated in $\mathcal{A}'$ if there is some finite basis B such all infinite fundamental antichains in $\mathrm{Av}(B)$ are equivalent (in $\mathcal{A}'$) to A.

Conjectures

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Not all minimal antichains are isolated.

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Conjecture

For each isolated antichain A "in" $\mathcal{A}'$, there is an algorithm to decide whether an arbitrary permutation belongs to $\mathrm{Cl}(A) \setminus A$.

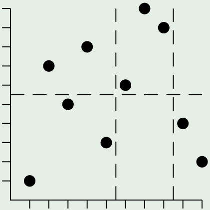
- Minimal isolated antichains have some kind of reliable structure.

Grid Classes

- Matrix $\mathcal{M}$ whose entries are permutation classes.
- $\mathrm{Grid}(\mathcal{M})$ the grid class of $\mathcal{M}$: all permutations which can be "gridded" so each cell satisfies constraints of $\mathcal{M}$.

Example

- Let $\mathcal{M} = \begin{pmatrix} \mathrm{Av}(21) & \mathrm{Av}(231) & \emptyset \\ \mathrm{Av}(123) & \emptyset & \mathrm{Av}(12) \end{pmatrix}$.

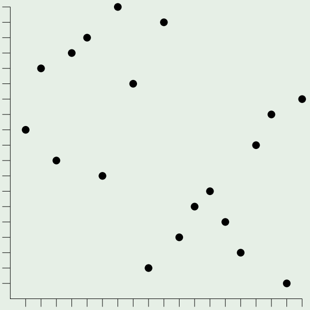


$\in \mathrm{Grid}(\mathcal{M})$

- Special case: all cells of $\mathcal{M}$ are $\mathrm{Av}(21)$ or $\mathrm{Av}(12)$.
- Rewrite $\mathcal{M}$ as a matrix with entries in $\{0, 1, -1\}$.

Example

$$\mathcal{M} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$



Monotone Grid Classes

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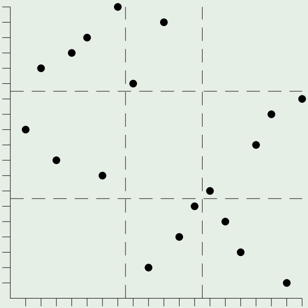

The Graph of a Matrix

- Graph of a matrix, $G(\mathcal{M})$, formed by connecting together all non-zero entries that share a row or column and are not "separated" by any other nonzero entry.

Example

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

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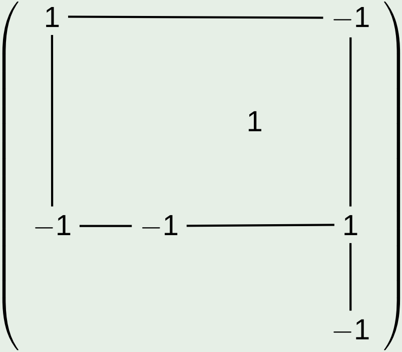

Monotone Grids and Partial Well Order

Theorem (Murphy and Vatter, 2003)

The monotone grid class $\mathrm{Grid}(\mathcal{M})$ *is pwo if and only if* $G(\mathcal{M})$ *is a forest, i.e.* $G(\mathcal{M})$ *contains no cycles.*

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Proof.

($\Leftarrow$) New shorter proof in Waton's Thesis (2007).

1	-1	
	1	
-1		-1

$\square$

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Proof.

($\Leftarrow$) Partial multiplication table.

-1	1	-1	
1		1	
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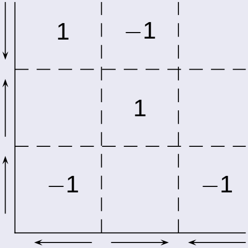
($\Leftarrow$) ± 1 correspond to directions.



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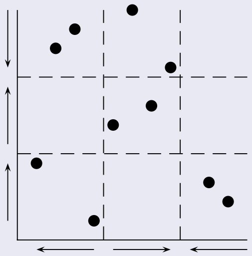
Proof.

($\Leftarrow$) Form one order per arrow.



- $1 < 9 < 8 < 4$.
- $5 < 10 < 6 < 7$.
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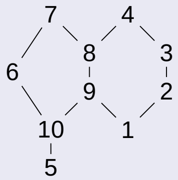
Proof.

$(\Leftarrow)$ No cycles, so this gives a poset.



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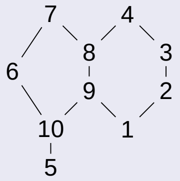
($\Leftarrow$) Linear extension: $5 < 10 < 1 < 9 < 2 < 6 < 8 < 3 < 7 < 4$



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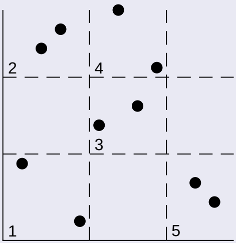

- Encode by region: 3412532541.

□

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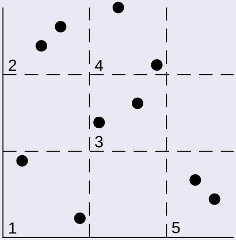

- Encode by region: 3412532541.
- Higman's Theorem: $\{1, 2, 3, 4, 5\}^*$ is pwo under the subword order.
- Encoding is reversible, hence $\mathrm{Grid}(\mathcal{M})$ is pwo.

□

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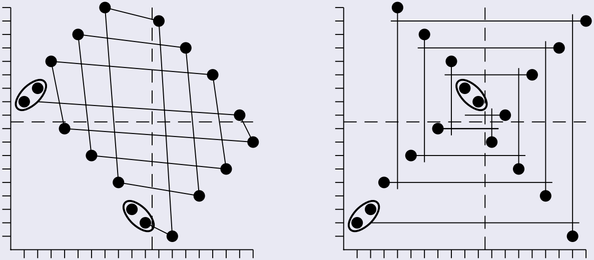
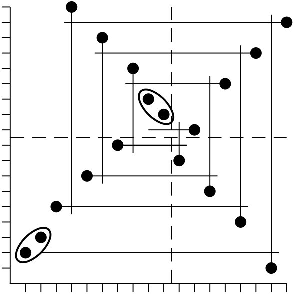
Proof.

(⇒) Construct fundamental antichains that “walk” around a cycle.



□

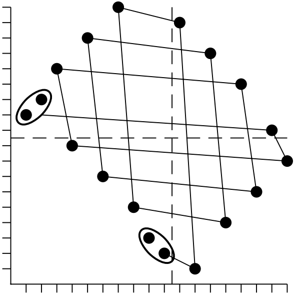
The Widdershins Antichain



- “Spirals” out from the centre.
- Constructed by means of a pin sequence.
- In general: a pin sequence with first and last pins inflated forms a fundamental antichain.

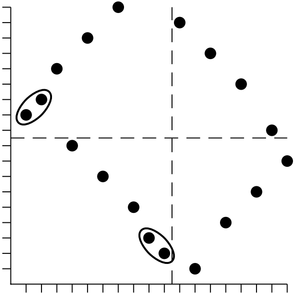
▸ Pin Sequences

Quasi-Square



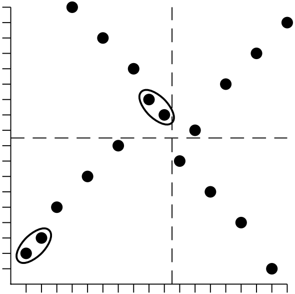
- Not constructible by a pin sequence.

Quasi-Square



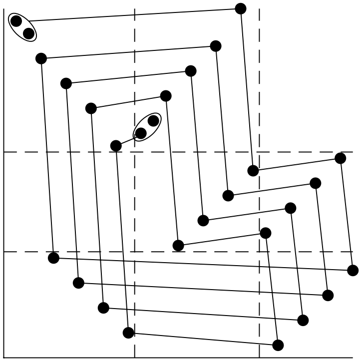
- Not constructible by a pin sequence.
- Flip each column...

Quasi-Square



- Not constructible by a pin sequence.
- ...Widdershins!

Bigger Grids



- Carry out row flips and column reversals: $r_1 \circ r_2 \circ r_3 \circ f_3$.

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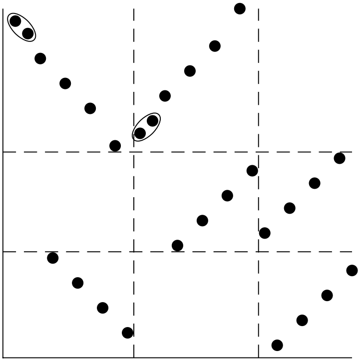

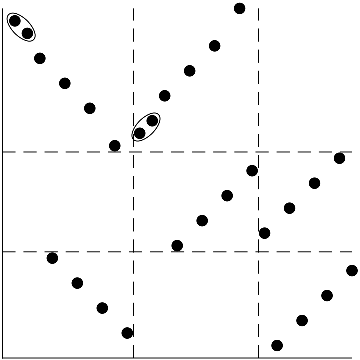

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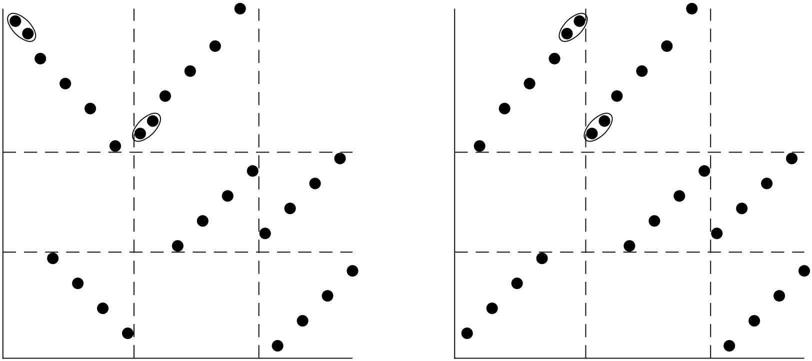

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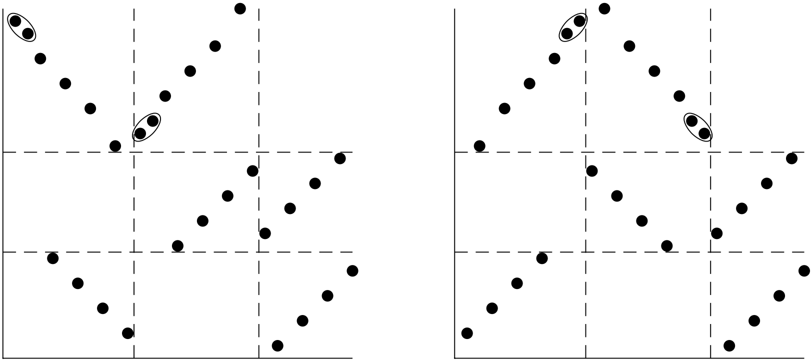

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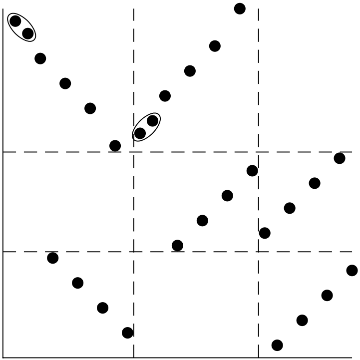

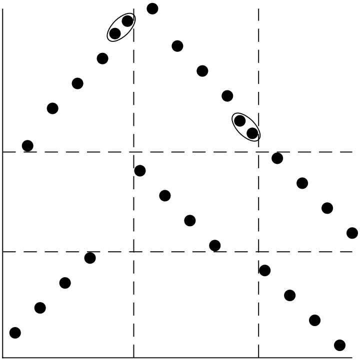

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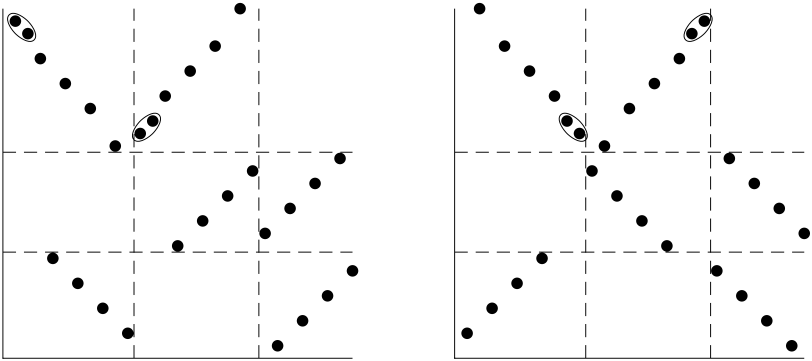

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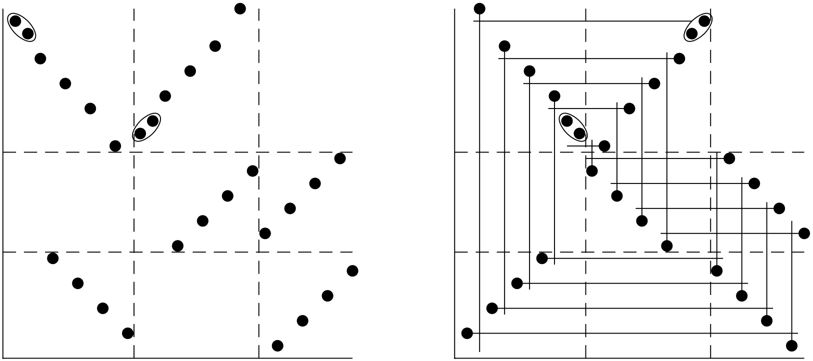

- Carry out row flips and column reversals: $r_1 \circ r_2 \circ r_3 \circ f_3$.
- Resulting structure behaves a bit like a pin sequence.

Grid Pin Sequences

- Local separation: p_{i+1} separates p_i from p_{i+1}.
- Row-column agreement: p_{i+1} must be placed in the same row or column as p_i.
- Local externality: p_{i+1} extends from $\mathrm{Rect}(p_{i-1}, p_i)$.
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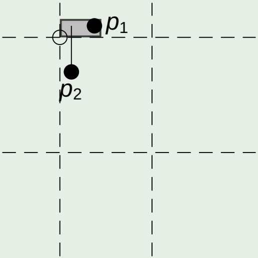
Example



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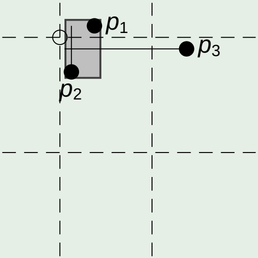

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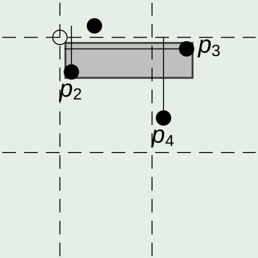

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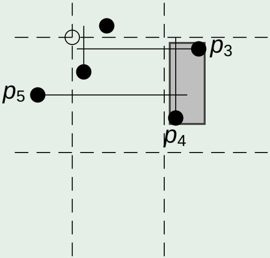

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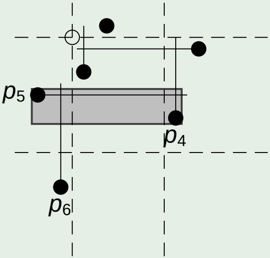

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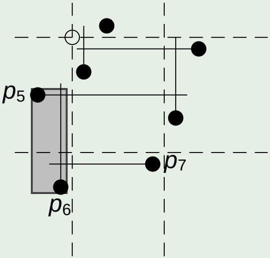

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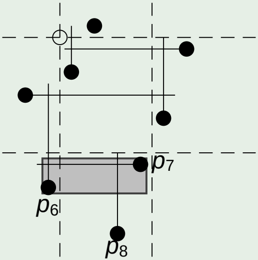

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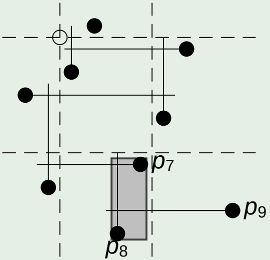

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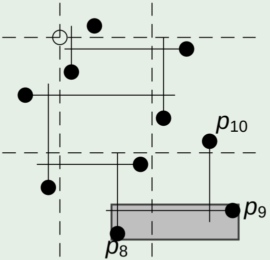

Optimism

- Grid pin sequences on an $m \times n$ grid can be encoded in a regular language on $\{c_1, \ldots, c_m, r_1, \ldots, r_n\}$.

▸ Grid Pin Encoding

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Conjecture

It is decidable whether a subclass of monotone grid class ("monotone griddable") given by a finite basis is partially well ordered.

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Conjecture

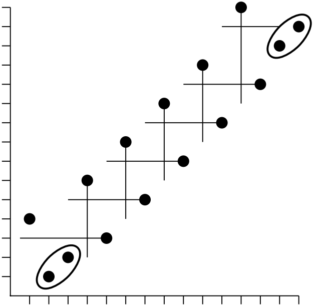
It is decidable whether a subclass of monotone grid class ("monotone griddable") given by a finite basis is partially well ordered.

Theorem (Huczynska and Vatter, 2006)

A permutation class is monotone griddable if and only if it does not contain arbitrarily long sums of 21 or skew sums of 12.

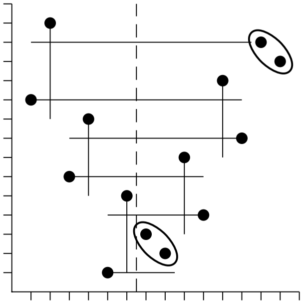
Other Antichains

- Increasing Oscillating — pin sequence in a single cell.



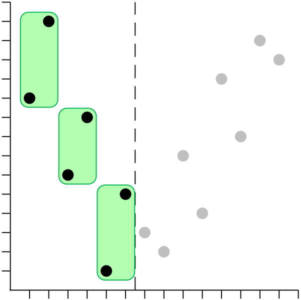
Other Antichains

- Two cells: antichain V.



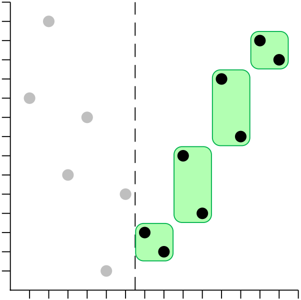
Other Antichains

- Two cells: antichain V.
- LHS: skew sums of 12.



Other Antichains

- Two cells: antichain V.
- RHS: direct sums of 21.



Juxtaposition

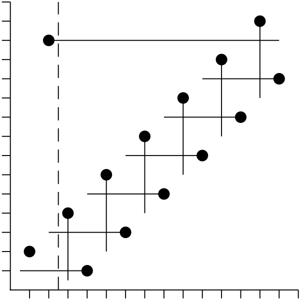
- The juxtaposition of two classes $\mathcal{C}$ and $\mathcal{D}$ is $[\mathcal{C}\ \mathcal{D}] = \mathrm{Grid}(\mathcal{C}\ \mathcal{D})$.
- Think of them as grid classes with 2 cells.

Question

When is the juxtaposition of two classes pwo?

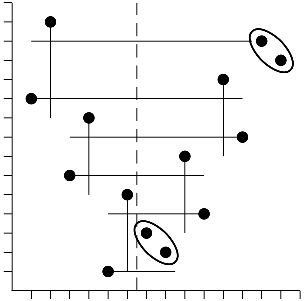
Juxtaposition II

- If $\mathcal{D}$ contains arbitrarily long oscillations and $\mathcal{C} \neq \mathrm{Av}(12, 21)$ then $[\mathcal{C}\ \mathcal{D}]$ is not pwo. ("Tied by One" antichain)



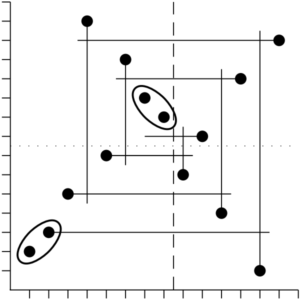
Juxtaposition II

- If $\mathcal{C}$ and $\mathcal{D}$ both contain arbitrarily long sums of 21 or skew sums of 12, then $[\mathcal{C}\ \mathcal{D}]$ is not pwo.



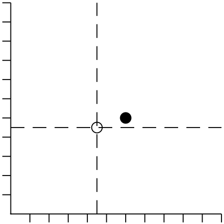
Juxtaposition II

- If $\mathcal{C}$ and $\mathcal{D}$ do not contain arbitrarily long sums of 21 or skew sums of 12, then they are monotone griddable.
- Not pwo if $\mathcal{C}$ and $\mathcal{D}$ contain arbitrarily long vertical alternations.

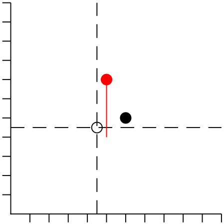


Thanks!

Appendix: Proper Pin Sequences



- Start with a point placed in relation to the origin.



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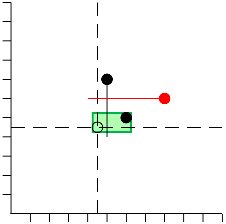

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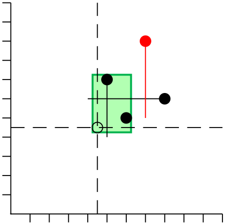

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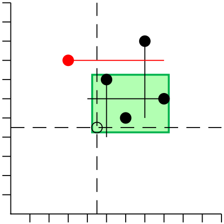

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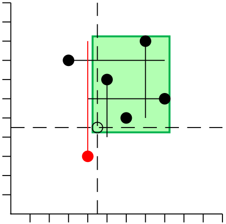

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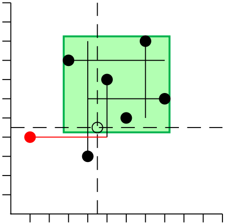

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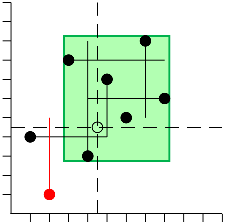

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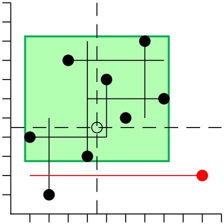

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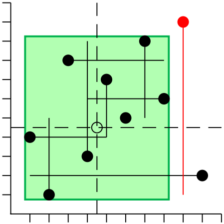

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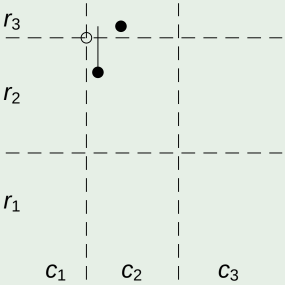
Return

Encoding Grid Pin Sequences

- Letter r_i: place a pin in row i.
- Letter c_j: place a pin in column j.
- This defines the placement of the pin uniquely.
- For example: r_2

Example



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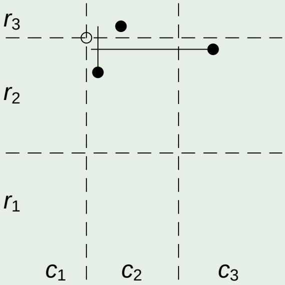

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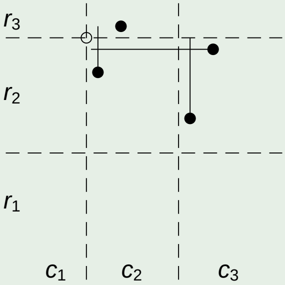

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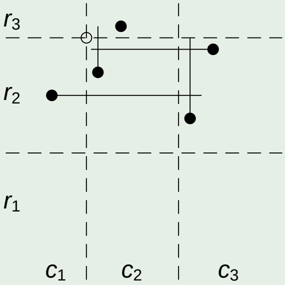

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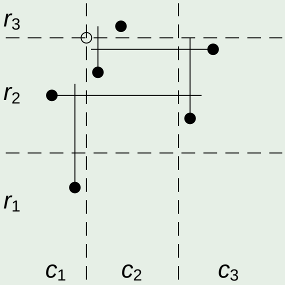

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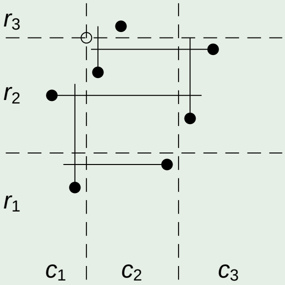

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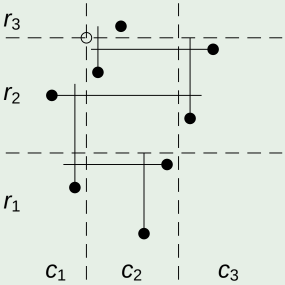

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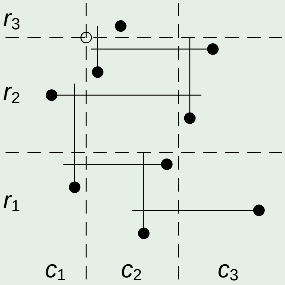

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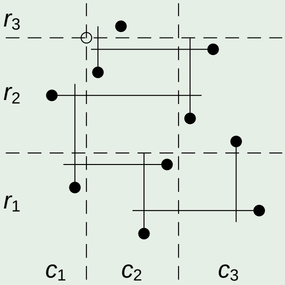
- Letter r_i: place a pin in row i.
- Letter c_j: place a pin in column j.
- This defines the placement of the pin uniquely.
- For example: $r_2 c_3 r_2 c_1 r_1 c_2 r_1 c_3 r_2$

Example



Return