Permutation classes and infinite antichains

Robert Brignall

Based on joint work with David Bevan and Nik Ruškuc

Dartmouth College, 12th July 2018

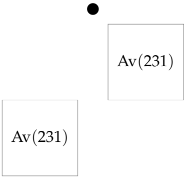
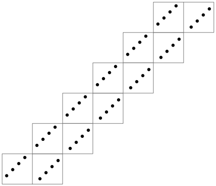
Typical questions in PP

For a permutation class $\mathcal{C}$:

- What is the growth rate?
- What is the generating function? (e.g. rational, algebraic, D-finite)
- What is the basis? (Is it finite?)
- What do the permutations 'look like'?

Examples: Av(231) and Av(321)

Both enumerated by Catalan numbers: $f(z) = \dfrac{1-\sqrt{1-4z}}{2z}$

	Av(231)	Av(321)
Growth rate	4	4
Generating function	algebraic	algebraic
Basis	231	321
'Look like'		

What about *subclasses* of Av(231), Av(321)?

	$\mathcal{C} \subsetneq \text{Av}(231)$	$\mathcal{D} \subsetneq \text{Av}(321)$
Growth rate		
Generating function		
Basis		

What about *subclasses* of Av(231), Av(321)?

	$\mathcal{C} \subsetneq \mathrm{Av}(231)$	$\mathcal{D} \subsetneq \mathrm{Av}(321)$
Growth rate	Countably many possibilities	Includes $[2.36, 2.48]$ (Bevan, 2018)
Generating function		
Basis		

What about *subclasses* of Av(231), Av(321)?

	$\mathcal{C} \subsetneq \mathrm{Av}(231)$	$\mathcal{D} \subsetneq \mathrm{Av}(321)$
Growth rate	Countably many possibilities	Includes $[2.36, 2.48]$ (Bevan, 2018)
Generating function	Rational (Albert, Atkinson, 2005)	Could be anything
Basis		

What about *subclasses* of Av(231), Av(321)?

	$\mathcal{C} \subsetneq \mathrm{Av}(231)$	$\mathcal{D} \subsetneq \mathrm{Av}(321)$
Growth rate	Countably many possibilities	Includes $[2.36, 2.48]$ (Bevan, 2018)
Generating function	Rational (Albert, Atkinson, 2005)	Could be anything
Basis	Finite	Finite or infinite

What's causing Av(321) to misbehave?

§1 An antichain is born

Back in 1972...

The Open University

Sorting Using Networks of Queues and Stacks

ROBERT TARJAN

Stanford University, Stanford, California*

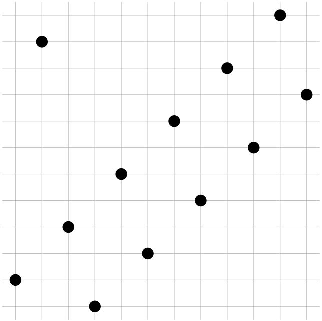
We might have conjectured that some finite set of patterns would characterize the sequences sortable in a switchyard. However, using the concept of the union graph, we may disprove this for the case of two parallel stacks:

LEMMA 6. *There is an infinite set of permutations, none of which contains another as a pattern, and such that each permutation is unsortable using two parallel stacks.*

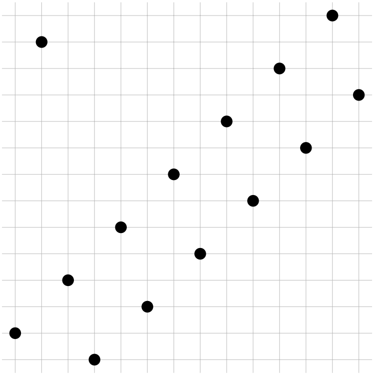
PROOF. Consider the union graphs of the permutations (2341) and (27416385) (see Figure 5).

We may generalize the second example in Figure 5 to give a permutation whose union graph is a cycle of length $2n + 1$, for arbitrary $n \geq 2$. $(2, 4n - 1, 4, 1, 6, 3, 8, 5, \cdots, 4n, 4n - 3)$ is the general permutation. Since the union graph of this permu-

An antichain is born

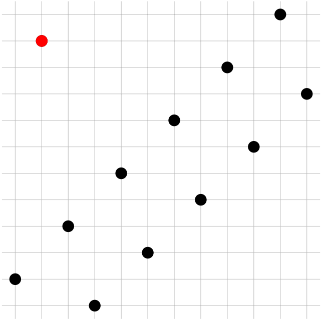


2 11 4 1 6 3 8 5 10 7 12 9

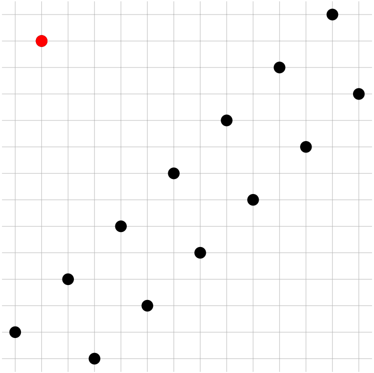


2 13 4 1 6 3 8 5 10 7 12 9 14 11

An antichain is born

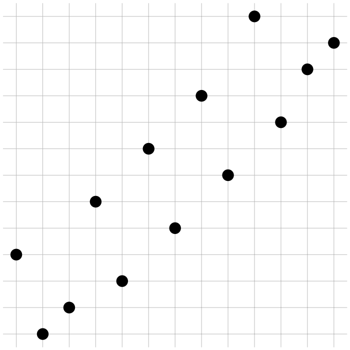


2 11 4 1 6 3 8 5 10 7 12 9

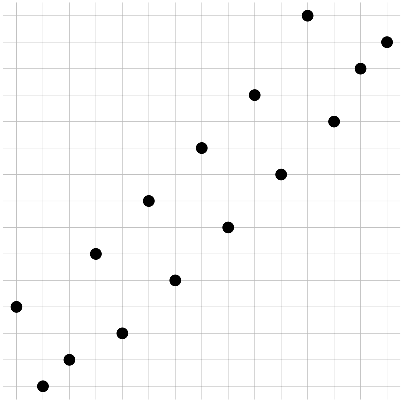


2 13 4 1 6 3 8 5 10 7 12 9 14 11

An antichain is born

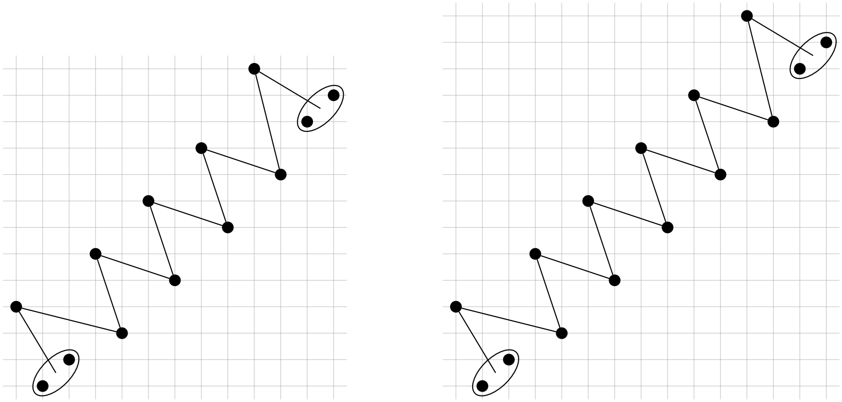


4 1 2 6 3 8 5 10 7 13 9 11 12



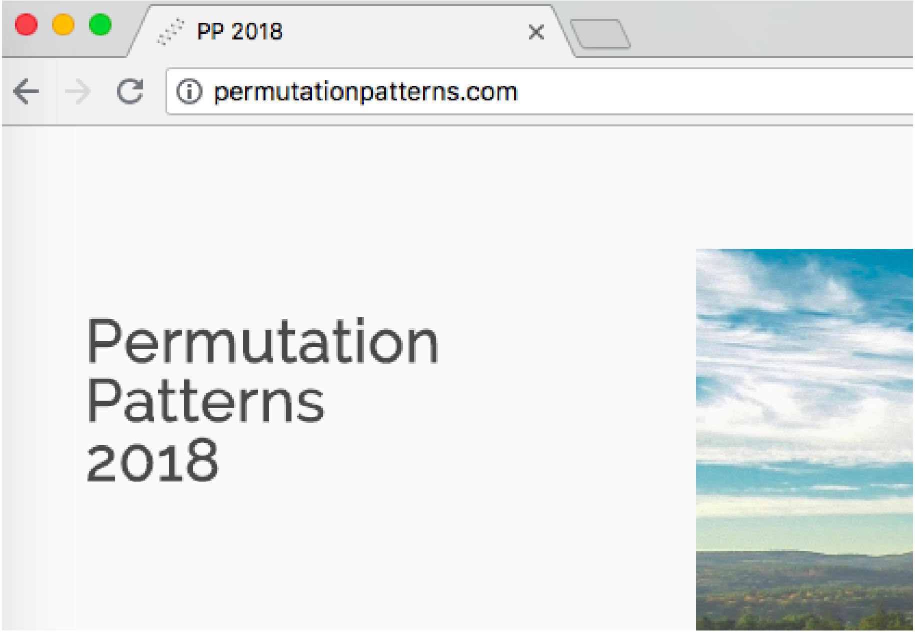
4 1 2 6 3 8 5 10 7 12 9 15 11 13 14

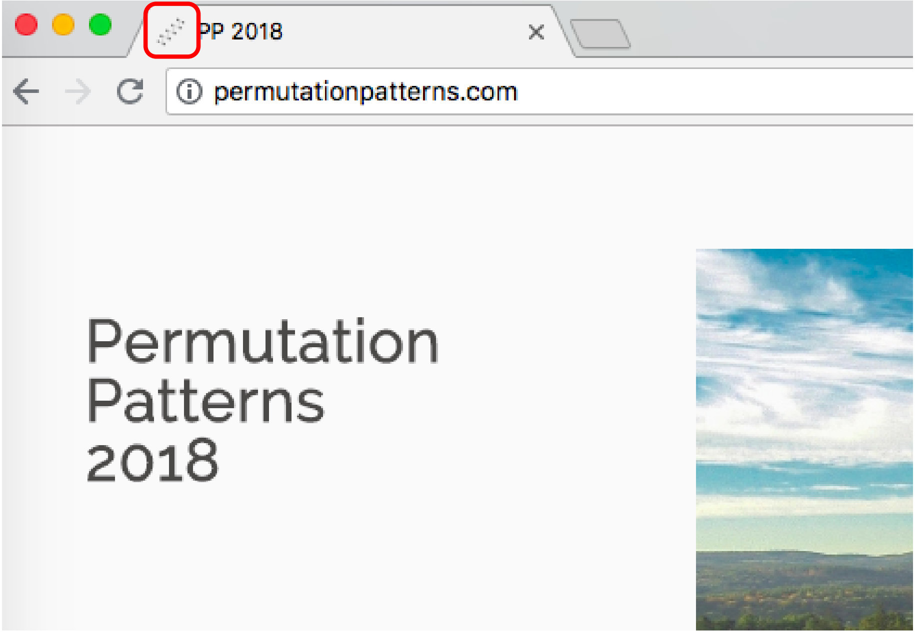
An antichain is born



4 1 2 6 3 8 5 10 7 13 9 11 12 $\not\leq$ 4 1 2 6 3 8 5 10 7 12 9 15 11 13 14

The increasing oscillating antichain, $\mathfrak{Osc}$, avoids 321


PP 2018
permutationpatterns.com
Permutation
Patterns
2018


PP 2018
permutationpatterns.com
Permutation
Patterns
2018

Formality, briefly

- An antichain: any set of permutations where no permutation is contained in any other.

$$\mathfrak{A} = \{\alpha_1, \alpha_2, \ldots : \alpha_i \not\leq \alpha_j \text{ for all } i, j\}.$$

- N.B. By minimality, the basis of a permutation class is always an antichain.
- Well-quasi-order or partial well-order: no infinite antichains. [I'll avoid these terms in this talk.]

Antichains, bases and enumeration

Proposition (Atkinson, Murphy, Ruškuc, 2002)

Let $\mathcal{C}$ be a finitely based permutation class. The following are equivalent:

(1) *Every subclass of $\mathcal{C}$ is finitely based,*

(2) *$\mathcal{C}$ contains at most countably many subclasses,*

(3) *$\mathcal{C}$ has no infinite antichain.*

Conjecture (Vatter, 2015)

If $\mathcal{C}$ is a permutation class that contains no infinite antichains, then it has an algebraic generating function.

Back to those subclasses of $\mathrm{Av}(231)$, $\mathrm{Av}(321)$

	$\mathcal{C} \subsetneq \mathrm{Av}(231)$	$\mathcal{D} \subsetneq \mathrm{Av}(321)$
Growth rate	Countably many possibilities	Includes $[2.36, 2.48]$ (Bevan, 2018)
Generating function	Rational (Albert, Atkinson, 2005)	Could be anything
Basis	Finite	Finite or infinite
Infinite antichains	None	$\mathfrak{Osc}$

Diversion 1: intervals of growth rates

Theorem (Bevan, 2018; Vatter, 2010)

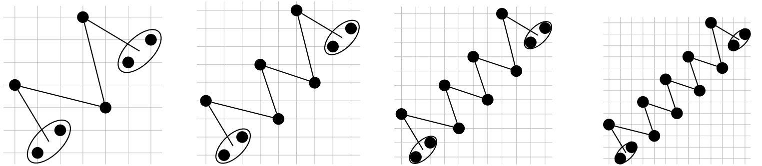
Every real number above 2.35698 *is the growth rate of some permutation class.*

'Proof'.

Create sum-closed classes by choosing sum indecomposables from $\mathfrak{Osc}$, and some easy variants.
Using cleverness, find a class with any growth rate above the unique real root of $x^8 - 2x^7 - x^5 - x^4 - 2x^3 - 2x^2 - x - 1$. □

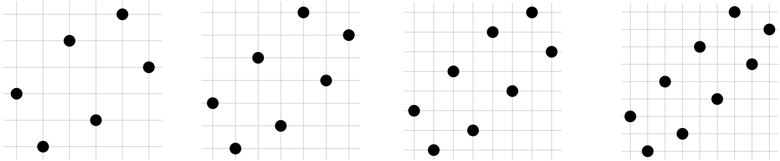
N.B. 'sum-closed' guarantees existence of a growth rate (Arratia, 1999)

Underpinning $\mathfrak{Osc}$



4 1 2 7 3 5 6   4 1 2 6 3 9 5 7 8   4 1 2 6 3 8 5 11 7 9 10   4 1 2 6 3 8 5 10 7 13 9 11 12

$\mathfrak{Osc}$ (above) and its ‘easy variants’ all build on increasing oscillations:



$$3\,1\,5\,2\,6\,4 \;\leq\; 3\,1\,5\,2\,7\,4\,6 \;\leq\; 3\,1\,5\,2\,7\,4\,8\,6 \;\leq\; 3\,1\,5\,2\,7\,4\,9\,6\,8$$

N.B. These form a chain (not an antichain!).

Diversion 2: Gollan permutations

SIAM J. COMPUT.
Vol. 25, No. 2, pp. 272–289, April 1996

003

GENOME REARRANGEMENTS AND SORTING BY REVERSALS*

VINEET BAFNA† AND PAVEL A. PEVZNER‡

Define $d(n) = \max_{\pi \in S_n} d(\pi)$ to be the *reversal diameter* of the symmetric group of order n. Gollan conjectured that $d(n) = n - 1$ and that only one permutation γ_n, and its inverse, γ_n^{-1}, require $n - 1$ reversals to be sorted (see Kececioglu and Sankoff [KS93] for details). The *Gollan* permutation, in one-line notation, is defined as follows:

$$\gamma_n = \begin{cases} (3, 1, 5, 2, 7, 4, \ldots, n-3, n-5, n-1, n-4, n, n-2), & n \text{ even}, \\ (3, 1, 5, 2, 7, 4, \ldots, n-6, n-2, n-5, n, n-3, n-1), & n \text{ odd}. \end{cases}$$

Diversion 3: Rational superclasses

Theorem (Albert, B., Vatter, 2013)

Every proper permutation class $\mathcal{C}$ is contained in a permutation class with a rational generating function.

'Proof'.

Use increasing oscillations to make an enormous infinite antichain

Diversion 3: Rational superclasses

Theorem (Albert, B., Vatter, 2013)

Every proper permutation class $\mathcal{C}$ is contained in a permutation class with a rational generating function.

'Proof'.

Use increasing oscillations to make an enormous infinite antichain

$$\mathfrak{A}$$

Diversion 3: Rational superclasses

Theorem (Albert, B., Vatter, 2013)

Every proper permutation class $\mathcal{C}$ is contained in a permutation class with a rational generating function.

'Proof'.

Use increasing oscillations to make an enormous infinite antichain

$$\mathfrak{A}$$

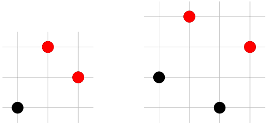
such that $\mathrm{Av}(\mathfrak{A}) \cup \mathfrak{A}$ has a rational generating function.
Union this with $\mathcal{C}$, and remove enough antichain elements of each length to preserve rationality. □

§2 Labelled containment

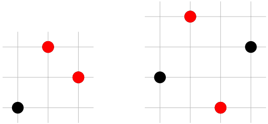
Labelled containment

- Colour/label each entry of σ and π red or black.
- $\sigma \leq_{\ell} \pi$ if σ embeds in π so that the labels match up. (Generalisations with more labels possible.)

Examples:

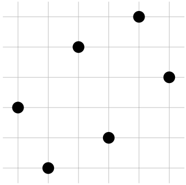


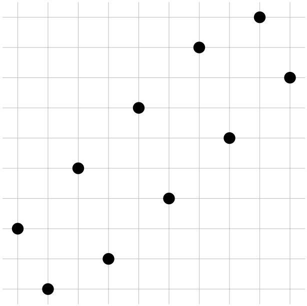
$1\,3\,2 \quad \leq_{\ell} \quad 2\,4\,1\,3$



$1\,3\,2 \quad \not\leq_{\ell} \quad 2\,4\,1\,3$

Labelled containment and $\mathfrak{Osc}$

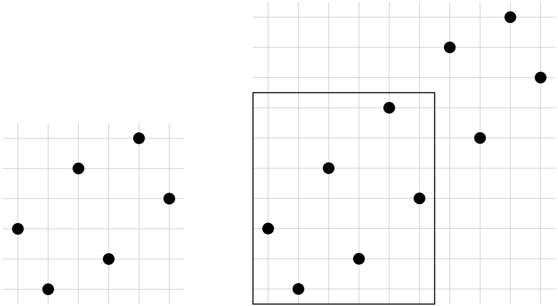




3 1 5 2 6 4 $\leq$ 3 1 5 2 7 4 9 6 10 8

- Increasing oscillations only embed contiguously

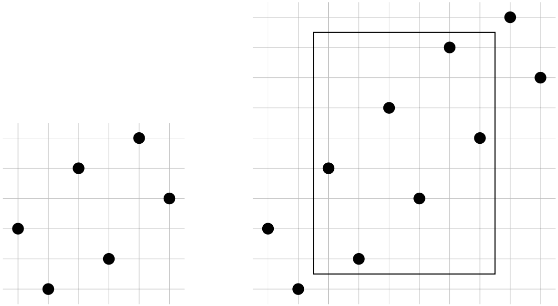
Labelled containment and $\mathfrak{Osc}$



3 1 5 2 6 4 $\leq$ 3 1 5 2 7 4 9 6 10 8

- Increasing oscillations only embed contiguously

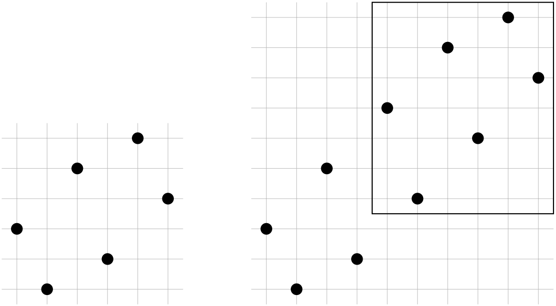
Labelled containment and $\mathfrak{Osc}$



$$3\,1\,5\,2\,6\,4 \quad \leq \quad 3\,1\,5\,2\,7\,4\,9\,6\,10\,8$$

- Increasing oscillations only embed contiguously

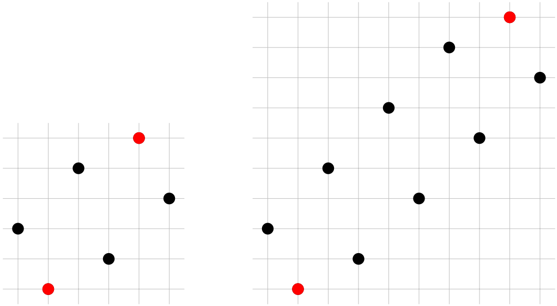
Labelled containment and $\mathfrak{Osc}$



3 1 5 2 6 4 $\leq$ 3 1 5 2 7 4 9 6 10 8

- Increasing oscillations only embed contiguously

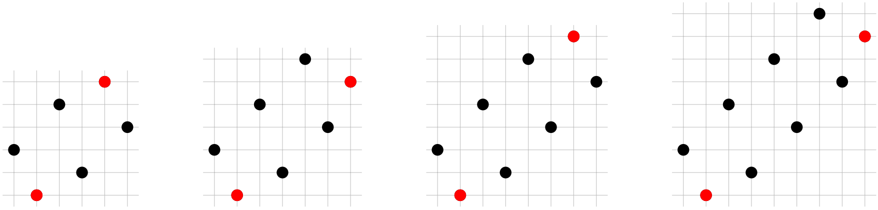
Labelled containment and $\mathfrak{Osc}$



3 1 5 2 6 4 $\not\leq_\ell$ 3 1 5 2 7 4 9 6 10 8

- Cannot embed lowest & highest into lowest & highest

A labelled infinite antichain



3 1 5 2 6 4   3 1 5 2 7 4 6   3 1 5 2 7 4 8 6   3 1 5 2 7 4 9 6 8

Warning! Abuse of notation:

> *'A class $\mathcal{C}$ contains a labelled infinite antichain'*

really means

> *'$\mathcal{C}$ contains an infinite set of permutations whose entries can be labelled red/black so that it forms an infinite antichain in $\leq_\ell$.'*

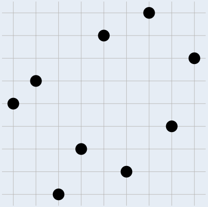
No labelled antichain ⇒ finite basis

Proposition (After Pouzet, 1972)

A permutation class $\mathcal{C}$ that contains no infinite labelled antichain is finitely based.

Proof.

Suppose $\mathcal{C} = \mathrm{Av}(\mathfrak{B})$ is not finitely based. For each $\beta \in \mathfrak{B}$:



β

□

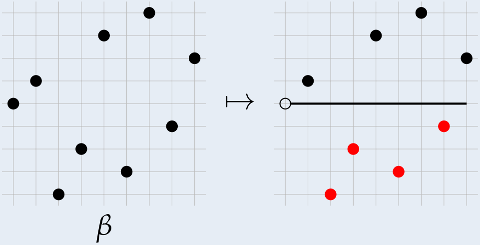
No labelled antichain $\Rightarrow$ finite basis

Proposition (After Pouzet, 1972)

A permutation class $\mathcal{C}$ that contains no infinite labelled antichain is finitely based.

Proof.

Suppose $\mathcal{C} = \mathrm{Av}(\mathfrak{B})$ is not finitely based. For each $\beta \in \mathfrak{B}$:



$\square$

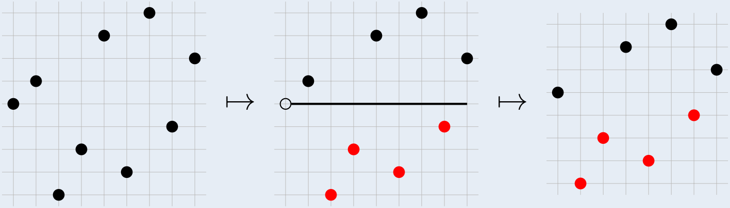
No labelled antichain $\Rightarrow$ finite basis

Proposition (After Pouzet, 1972)

A permutation class $\mathcal{C}$ that contains no infinite labelled antichain is finitely based.

Proof.

Suppose $\mathcal{C} = \text{Av}(\mathfrak{B})$ is not finitely based. For each $\beta \in \mathfrak{B}$:



β $\qquad$ β^-

$\mathfrak{B}^- = \{\beta^- : \beta \in \mathfrak{B}\}$ is a labelled antichain in $\mathcal{C}$: contradiction. □

Back to those subclasses of Av(321)

Theorem (Albert, B., Ruškuc, Vatter)

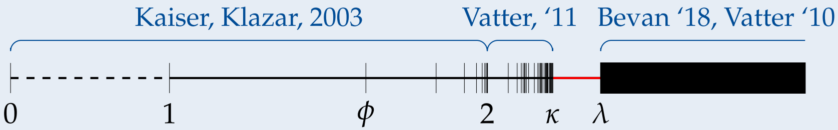
If $\mathcal{D} \subsetneq Av(321)$ is finitely based or does not contain an infinite antichain, then it has a rational generating function.

Furthermore, if $\mathcal{D} \subsetneq Av(321)$ contains a (labelled or unlabelled) infinite antichain then it contains long increasing oscillations.

Thus, if $\mathcal{D} \subsetneq \text{Av}(321)$ avoids some oscillation, then $\mathcal{D}$ is finitely based and has a rational generating function.

Small permutation classes

What growth rates are allowed?



- Growth rates below $\kappa \approx 2.20557$: no (labelled or unlabelled) infinite antichains ($\Rightarrow$ all classes finitely based).
- At κ: increasing oscillations and $\mathfrak{Osc}$ appear ($\Rightarrow$ uncountably many classes).

Theorem (Albert, Ruškuc, Vatter, 2015)

Every permutation class $\mathcal{C}$ with $gr(\mathcal{C}) < \kappa$ has a rational generating function.

Smallish permutation classes

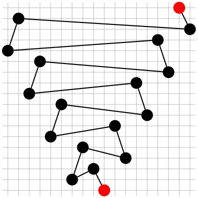
- $\xi \approx 2.30522$ marks another phase transition: from countably many to uncountably many different growth rates (Vatter).
- Classification of growth rates from κ to ξ (Pantone, Vatter).

Smallish permutation classes

- $\xi \approx 2.30522$ marks another phase transition: from countably many to uncountably many different growth rates (Vatter).
- Classification of growth rates from κ to ξ (Pantone, Vatter).

Conjecture

The next infinite (labelled) antichain first appears in a class of growth rate $\nu \approx 3.06918$.

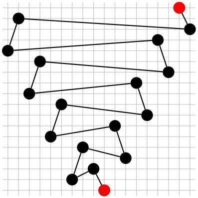


Smallish permutation classes

- $\xi \approx 2.30522$ marks another phase transition: from countably many to uncountably many different growth rates (Vatter).
- Classification of growth rates from κ to ξ (Pantone, Vatter).

Conjecture

The next infinite (labelled) antichain first appears in a class of growth rate $\nu \approx 3.06918$.



Conjollary

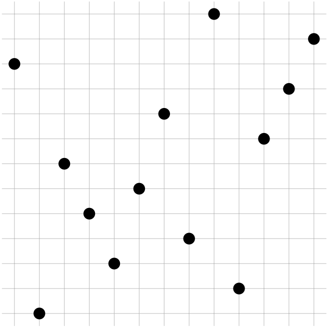
Any class $\mathcal{C}$ with $gr(\mathcal{C}) < 3.06918$ which contains only bounded length oscillations is finitely based.

§3 Grid classes

Grid classes

- $\mathcal{M}$ a $0, \pm 1$ matrix.
- $\pi \in \text{Grid}(\mathcal{M})$ if π can be gridded so that each cell of π is $\left\{ \begin{array}{c} \text{empty} \\ \text{increasing} \\ \text{decreasing} \end{array} \right\}$ if the corresponding entry of $\mathcal{M}$ is $\left\{ \begin{array}{c} 0 \\ 1 \\ -1 \end{array} \right\}$.

$$\mathcal{M} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

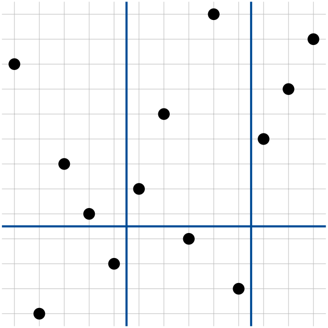


$$\pi = 11\ 1\ 7\ 5\ 3\ 6\ 9\ 4\ 13\ 2\ 8\ 10\ 12$$

Grid classes

- $\mathcal{M}$ a $0, \pm 1$ matrix.
- $\pi \in \mathrm{Grid}(\mathcal{M})$ if π can be gridded so that each cell of π is $\left\{ \begin{array}{c} \text{empty} \\ \text{increasing} \\ \text{decreasing} \end{array} \right\}$ if the corresponding entry of $\mathcal{M}$ is $\left\{ \begin{array}{c} 0 \\ 1 \\ -1 \end{array} \right\}$.

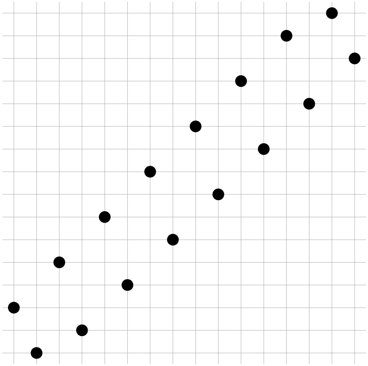
$$\mathcal{M} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$



$$\mathrm{Grid}(\mathcal{M}) \ni \pi = 11\ 1\ 7\ 5\ 3\ 6\ 9\ 4\ 13\ 2\ 8\ 10\ 12$$

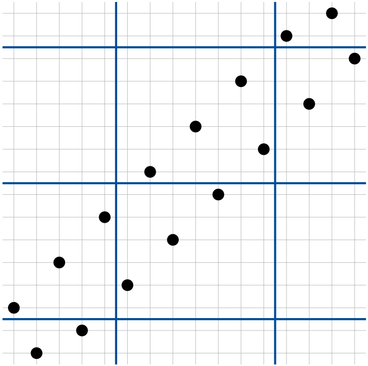
The antithesis of $\mathfrak{Osc}$

Long increasing oscillations don't grid nicely.



The antithesis of $\mathfrak{Osc}$

Long increasing oscillations don't grid nicely.



Those questions again

	Grid($\mathcal{M}$)
Growth rate	
Generating function	
Basis	
Infinite antichains	

Those questions again

	Grid($\mathcal{M}$)
Growth rate	$\rho(\mathcal{M})^2$ (Bevan, 2015)
Generating function	
Basis	
Infinite antichains	

Those questions again

	Grid($\mathcal{M}$)
Growth rate	$\rho(\mathcal{M})^2$ (Bevan, 2015)
Generating function	Rational if acyclic†, otherwise …
Basis	
Infinite antichains	

† – see Albert, Atkinson, Bouvel, Ruškuc, Vatter, 2013

Those questions again

	Grid($\mathcal{M}$)
Growth rate	$\rho(\mathcal{M})^2$ (Bevan, 2015)
Generating function	Rational if acyclic†, otherwise ...
Basis	Finite if acyclic†, otherwise ...
Infinite antichains	

† – see Albert, Atkinson, Bouvel, Ruškuc, Vatter, 2013

Those questions again

	Grid($\mathcal{M}$)
Growth rate	$\rho(\mathcal{M})^2$ (Bevan, 2015)
Generating function	Rational if acyclic†, otherwise …
Basis	Finite if acyclic†, otherwise …
Infinite antichains	None iff acyclic (Murphy, Vatter, 2003)

† – see Albert, Atkinson, Bouvel, Ruškuc, Vatter, 2013

Cycles in a grid class

$\mathcal{M}$: $m \times n$ matrix.

$G_{\mathcal{M}}$: bipartite graph, vertices $\{c_1, \ldots, c_m, r_1, \ldots, r_n\}$, with $c_i r_j \in E(G_{\mathcal{M}})$ if $M_{ij} \neq 0$.

Example

$$\mathcal{M} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$G_{\mathcal{M}}$:

r_1	-1	1	1
r_2	1	-1	
	c_1	c_2	c_3

Cycles in a grid class

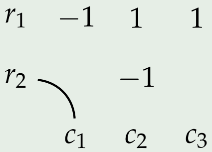
$\mathcal{M}$: $m \times n$ matrix.

$G_{\mathcal{M}}$: bipartite graph, vertices $\{c_1, \ldots, c_m, r_1, \ldots, r_n\}$, with $c_i r_j \in E(G_{\mathcal{M}})$ if $M_{ij} \neq 0$.

Example

$$\mathcal{M} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$G_{\mathcal{M}}$:



Cycles in a grid class

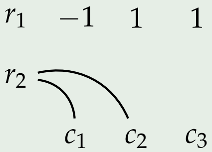
$\mathcal{M}$: $m \times n$ matrix.

$G_{\mathcal{M}}$: bipartite graph, vertices $\{c_1, \ldots, c_m, r_1, \ldots, r_n\}$, with $c_i r_j \in E(G_{\mathcal{M}})$ if $M_{ij} \neq 0$.

Example

$$\mathcal{M} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$G_{\mathcal{M}}$:



Cycles in a grid class

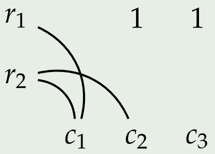
$\mathcal{M}$: $m \times n$ matrix.

$G_{\mathcal{M}}$: bipartite graph, vertices $\{c_1, \ldots, c_m, r_1, \ldots, r_n\}$, with $c_i r_j \in E(G_{\mathcal{M}})$ if $M_{ij} \neq 0$.

Example

$$\mathcal{M} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$G_{\mathcal{M}}$:



Cycles in a grid class

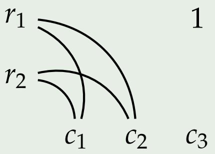
$\mathcal{M}$: $m \times n$ matrix.

$G_{\mathcal{M}}$: bipartite graph, vertices $\{c_1, \ldots, c_m, r_1, \ldots, r_n\}$, with $c_i r_j \in E(G_{\mathcal{M}})$ if $M_{ij} \neq 0$.

Example

$$\mathcal{M} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$G_{\mathcal{M}}$:



Cycles in a grid class

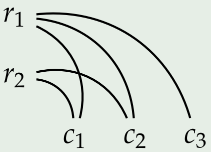
$\mathcal{M}$: $m \times n$ matrix.

$G_{\mathcal{M}}$: bipartite graph, vertices $\{c_1, \ldots, c_m, r_1, \ldots, r_n\}$, with $c_i r_j \in E(G_{\mathcal{M}})$ if $M_{ij} \neq 0$.

Example

$$\mathcal{M} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$G_{\mathcal{M}}$:



- Acyclic: $G_{\mathcal{M}}$ has no cycles.
- Unicyclic: $G_{\mathcal{M}}$ has at most one cycle.

Questions by cyclicity

Grid($\mathcal{M}$)	acyclic	unicyclic	polycyclic
Growth rate	$\rho(\mathcal{M})^2$	$\rho(\mathcal{M})^2$	$\rho(\mathcal{M})^2$
Generating function	Rational	Erm…	Erm…
Basis	Finite	Erm…	Erm…
Infinite antichains	None‡	Some…	Some…

‡ Not even labelled infinite antichains ($\Rightarrow$ finitely based).

Unicyclic grids

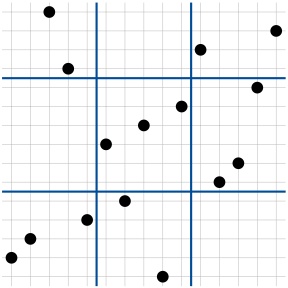
Following the role of $\mathfrak{Osc}$ in $\mathrm{Av}(321)$, we ask:

Question

When does a subclass $\mathcal{C}$ of a unicyclic grid class $\mathrm{Grid}(\mathcal{M})$ contain an infinite labelled antichain?

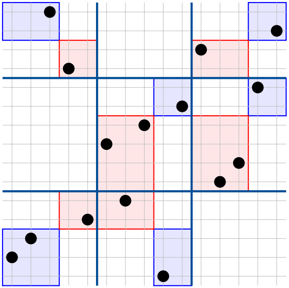
Key idea: ⊞-decomposition

Example: Let $\mathcal{M} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$



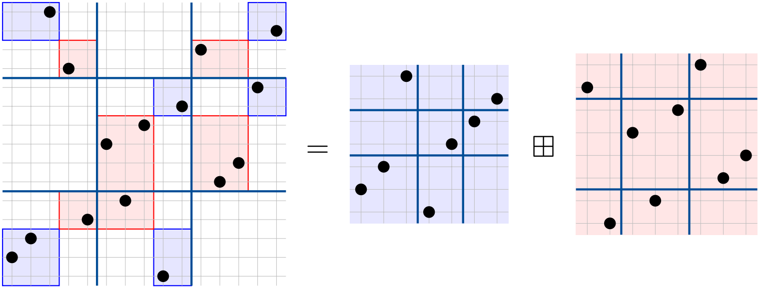
Key idea: ⊞-decomposition

Example: Let $\mathcal{M} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$



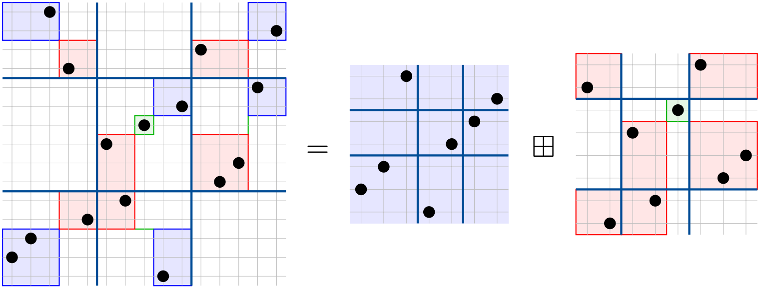
Key idea: ⊞-decomposition

Example: Let $\mathcal{M} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$



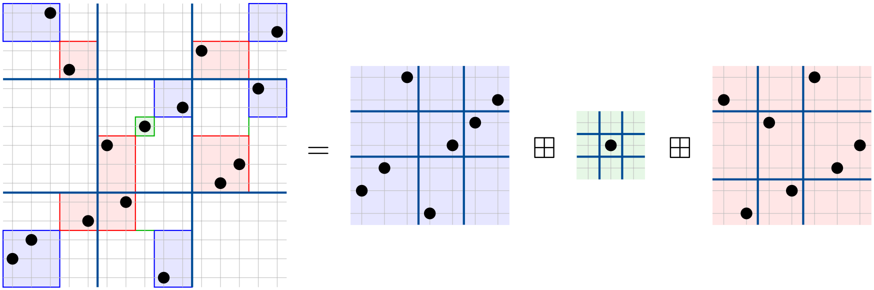
Key idea: ⊞-decomposition

Example: Let $\mathcal{M} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$



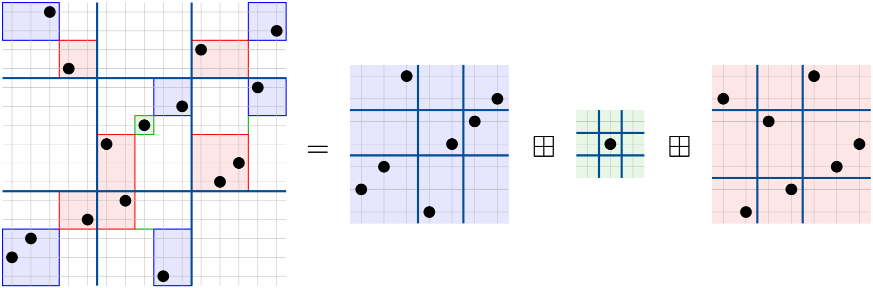
Key idea: $\boxplus$-decomposition

Example: Let $\mathcal{M} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$



Key idea: ⊞-decomposition

Example: Let $\mathcal{M} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$



Lemma

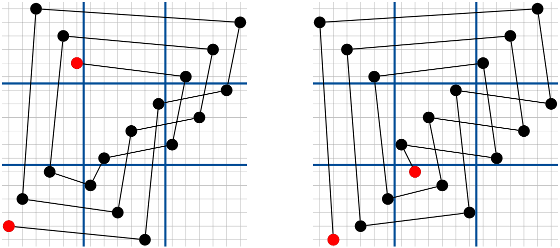
The class $\mathcal{C} \subseteq Grid(\mathcal{M})$ contains infinite labelled antichains if and only if the ⊞-indivisible gridded permutations in $\mathcal{C}$ do.

Consequences

Theorem (Bevan, B., Ruškuc)

For a unicyclic grid class Grid$(\mathcal{M})$, any subclass $\mathcal{C} \subseteq$ Grid$(\mathcal{M})$ contains (labelled) infinite antichains if and only if $\mathcal{C}$ has infinite intersection with one of two labelled antichains (per component of $G_{\mathcal{M}}$).

Example: For $\mathcal{M} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, they are:



Basis of unicyclic grids

Theorem (Bevan, B., Ruškuc)

Unicyclic grid classes are finitely based.

Proof outline (if it survives writing up…)

If $\mathcal{C} \subsetneq \mathrm{Grid}(\mathcal{M})$ contains no labelled antichains, then neither does $\mathcal{C}^{+1}$.

$\square$

Basis of unicyclic grids

Theorem (Bevan, B., Ruškuc)

Unicyclic grid classes are finitely based.

Proof outline (if it survives writing up...)

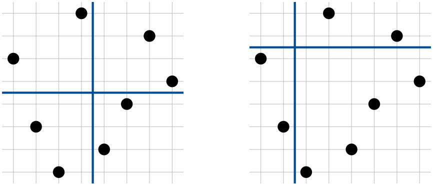
If $\mathcal{C} \subsetneq \mathrm{Grid}(\mathcal{M})$ contains no labelled antichains, then neither does $\mathcal{C}^{+1}$.

Write $\mathrm{Grid}(\mathcal{M}) = \mathrm{Av}(\mathfrak{B})$ and argue that $\mathfrak{B} \subset \mathcal{C}^{+1}$. $\square$

Enumeration of unicycles

Proposition (Bevan, PhD thesis 2015)

The gridded permutations in a unicyclic grid class have an algebraic generating function.



Work in progress

A unicyclic grid class *should* have an algebraic generating function.

Questions by cyclicity

Grid($\mathcal{M}$)	acyclic	unicyclic	polycyclic
Growth rate	$\rho(\mathcal{M})^2$	$\rho(\mathcal{M})^2$	$\rho(\mathcal{M})^2$
Generating function	Rational	Erm…	Erm…
Basis	Finite	Erm…	Erm…
Infinite antichains	None‡	Some…	Some…

‡ Not even labelled infinite antichains ($\Rightarrow$ finitely based).

Questions by cyclicity

Grid($\mathcal{M}$)	acyclic	unicyclic	polycyclic
Growth rate	$\rho(\mathcal{M})^2$	$\rho(\mathcal{M})^2$	$\rho(\mathcal{M})^2$
Generating function	Rational	?Algebraic	Erm…
Basis	Finite	Finite	Erm…
Infinite antichains	None‡	'Two'	Some…

‡ Not even labelled infinite antichains ($\Rightarrow$ finitely based).

Thanks!