From permutations to graphs

well-quasi-ordering and infinite antichains

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Joint work with Atminas, Korpelainen, Lozin and Vatter

18th December 2014

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Orderings on Structures

- Pick your favourite family of combinatorial structures.
  E.g. graphs, permutations, tournaments, posets, . . .

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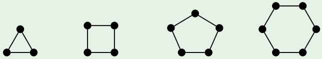
- Pick your favourite family of combinatorial structures.
  E.g. graphs, permutations, tournaments, posets, …
- Give your family an ordering.
  E.g. graph minor, induced subgraph, permutation containment, …
- Does your ordering contain infinite antichains?
  i.e. an infinite set of pairwise incomparable elements.

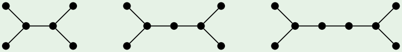
Example ((Induced) subgraph antichains)

Cycles:



…

"Split end" graphs:



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When are there antichains?

No infinite antichains = well-quasi-ordered.

- Words over a finite alphabet with subword ordering [Higman, 1952].
- Trees ordered by topological minors [Kruskal 1960; Nash-Williams, 1963]
- Graphs closed under minors [Robertson and Seymour, 1983—2004].

Infinite antichains.

- Graphs closed under induced subgraphs (or merely subgraphs).
- Permutations closed under containment.
- Tournaments, digraphs, posets, . . . with their natural induced substructure ordering.

Why study this?

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Algorithms inside well-quasi-ordered sets

- Polynomial-time recognition: is one graph a minor of another?
- Fixed-parameter tractability: e.g. graphs with vertex cover at most k can be recognised in polynomial time.

Miscellany

- Well-quasi-order = nice structure. Useful for other problems (e.g. enumeration)
- Connections with logic: Kruskal's Tree Theorem is unproveable in Peano arithmetic [Friedman, 2002]
- Antichains are pretty! (See later)
- It is fun [Kříž and Thomas, 1990]
- *Because it's there.* [Mallory]

Formal definition

- Quasi order: reflexive transitive relation.
- Partial order: quasi order + asymmetric.

Definition

Let $(S, \leq)$ be a quasi-ordered (or partially-ordered) set. Then S is said to be well quasi ordered (wqo) under $\leq$ if it

- is well-founded (no infinite descending chain), and
- contains no infinite antichain (set of pairwise incomparable elements).

- Well founded: usually trivial for finite combinatorial objects. This is all about the antichains.

My objects aren't wqo, what can I do?

- Don't panic! Maybe you could restrict to a subcollection?

Example: Cographs as induced subgraphs

$$\begin{aligned} \text{Cographs} &= \text{graphs containing no induced } P_4 \\ &= \text{closure of } K_1 \text{ under complementation and disjoint union.} \end{aligned}$$

- Cographs are well-quasi-ordered. [Damaschke, 1990]

- Learn to stop worrying and love the antichains! [sorry, Kubrick]

Downsets

Question

In your favourite ordering, which downsets contain infinite antichains?

- Downset (or hereditary property, or class): set $\mathcal{C}$ of objects such that
$$G \in \mathcal{C} \text{ and } H \leq G \text{ implies } H \in \mathcal{C}.$$

Examples

- Triangle-free graphs: downset under (induced) subgraphs. Not wqo.
- Cographs: downset under induced subgraphs. Wqo.
- Planar graphs: downset under graph minor. Wqo.
- Words over $\{0, 1\}$ with no '00' factor: downset under factor order. Not wqo: $010,\ 0110,\ 01110,\ 011110, \ldots$

Minimal forbidden elements

- Downsets often defined by the minimal forbidden elements.

Examples

- Triangle-free graphs: K_3 free as (induced) subgraph.
- Cographs: $\mathrm{Free}(P_4)$.
- Planar graphs: $\{K_5, K_{3,3}\}$-minor free graphs [Wagner's Theorem]
- Pattern-avoiding permutations: $\mathrm{Av}(321)$ (see later).

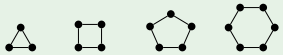
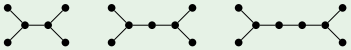
- Confusingly, the set of minimal forbidden elements is an antichain!
- Graph Minor Theorem $\Rightarrow$ every minor-closed class has finitely many forbidden elements.

Decision procedures

Question

In your favourite ordering, which downsets contain infinite antichains?

Known decision procedures

- Graph minors: no antichains anywhere!
- Subgraph order: a downset is wqo if and only if it contains neither  $\cdots$ nor  $\cdots$ [Ding, 1992]
- Factor order: downsets of words over a finite alphabet [Atminas, Lozin & Moshkov, 2013]

Theorem (Cherlin & Latka, 2000)

Any downset with k minimal forbidden elements is wqo iff it doesn't contain any of the infinite antichains in a finite collection Λ_k.

Plan for the rest of today

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Ordering of the day

Induced subgraph ordering, $H \leq_{\text{ind}} G$.

Question

For which m, n is the following true?

The set of permutation graphs with no induced P_m or K_n is wqo.

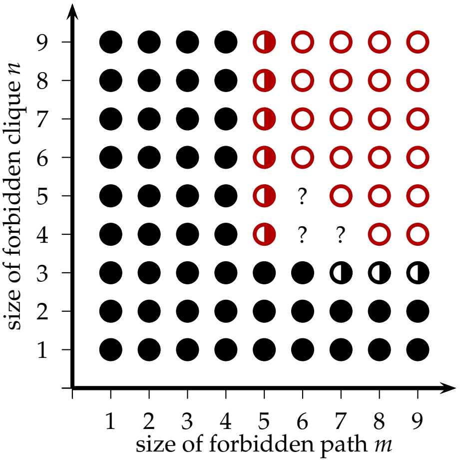
We'll:

- Build some antichains;
- Find structure to prove wqo.

Motivation?

- The 'right' level of difficulty: Interestingly complex, but tractable.
- Demonstration of some recently-developed structural theory.
- Expansion of the graph $\longleftrightarrow$ permutation interplay.

Forbidding paths and cliques

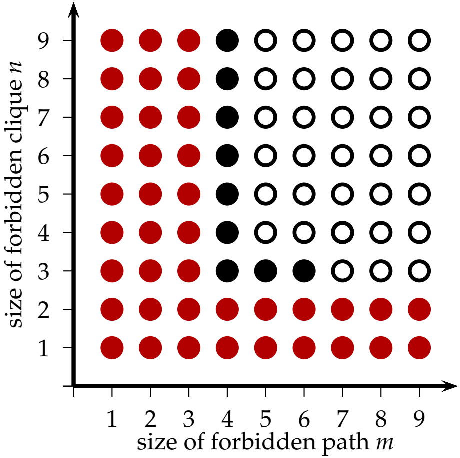


● = Graphs wqo

◑ = Permutation graphs wqo, graphs not wqo

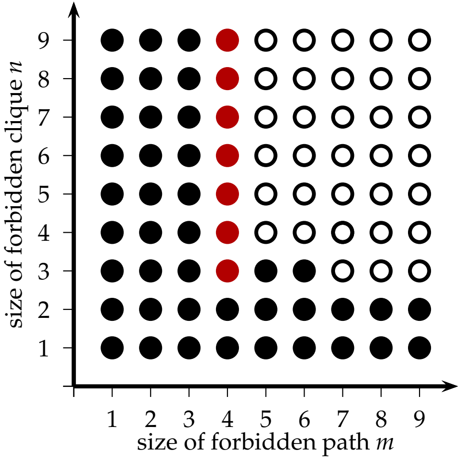
○ = Permutation graphs not wqo

~~Permutation~~ graphs



These classes are trivially wqo.

~~Permutation~~ graphs



Cographs are wqo [Damaschke, 1990]

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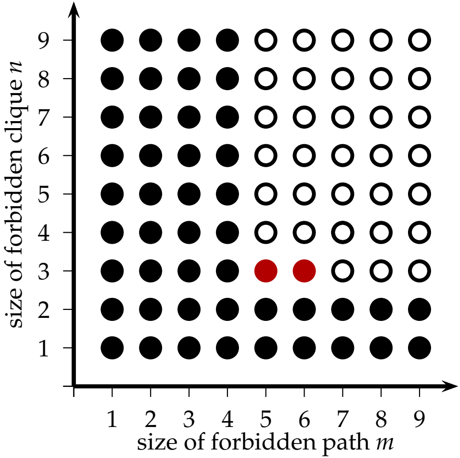
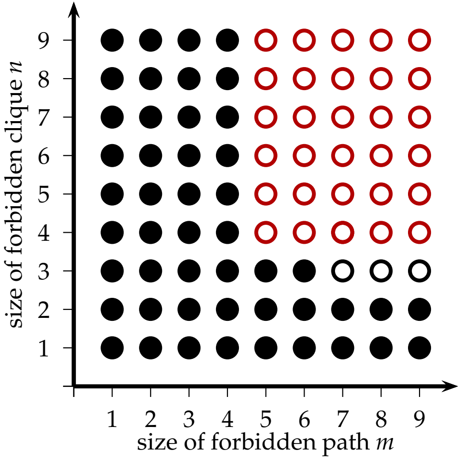


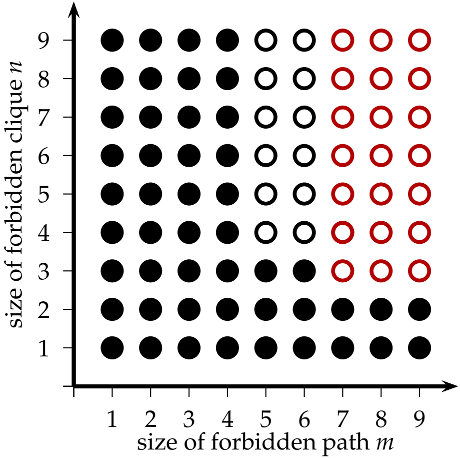
P_6, K_3-free graphs are wqo [Atminas and Lozin, 2014]

~~Permutation~~ graphs



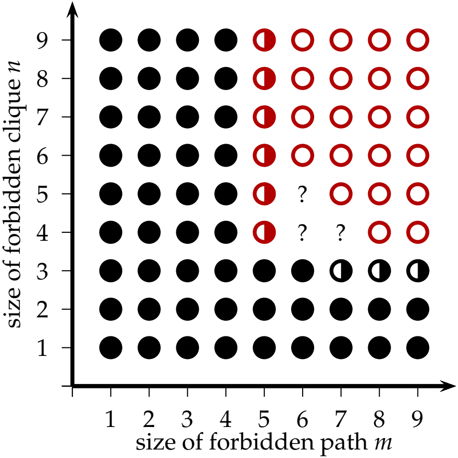
P_5, K_4-free graphs are not wqo [Korpelainen and Lozin, 2011]

~~Permutation~~ graphs



P_7, K_3-free graphs are not wqo [Korpelainen and Lozin, 2011b]

Forbidding paths and cliques



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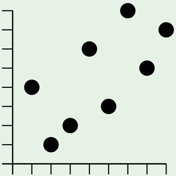
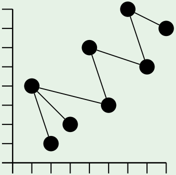
Permutation graphs



- Permutation $\pi = \pi(1) \cdots \pi(n)$
- Make a graph G_π: for $i < j$, $ij \in E(G_\pi)$ iff $\pi(i) > \pi(j)$.
- Note: $n \cdots 21$ becomes K_n.

Permutation graphs



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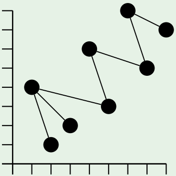
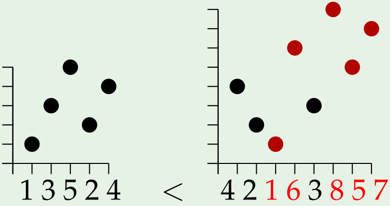


- Permutation graph = can be made from a permutation
  = comparability ∩ co-comparibility
  = comparability graphs of dimension 2 posets
- Lots of polynomial time algorithms here (e.g. MaxClique, TreeWidth)

Ordering permutations: containment

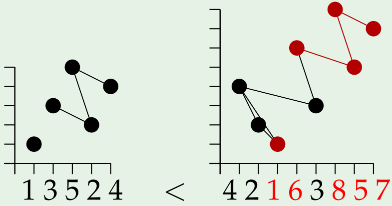
Example



- Pattern containment: a partial order, $\sigma \leq \pi$.

Ordering permutations: containment

Example



- Pattern containment: a partial order, $\sigma \leq \pi$.
- Draw the graphs: $G_\sigma \leq_{\text{ind}} G_\pi$.

Ordering permutations: containment

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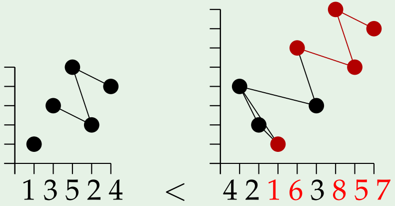


- Pattern containment: a partial order, $\sigma \leq \pi$.
- Draw the graphs: $G_\sigma \leq_{\text{ind}} G_\pi$.
- Permutation class: downset in this ordering:

$$\pi \in \mathcal{C} \text{ and } \sigma \leq \pi \text{ implies } \sigma \in \mathcal{C}.$$

- Avoidance: minimal forbidden permutation characterisation:

$$\mathcal{C} = \mathrm{Av}(B) = \{\pi : \beta \not\leq \pi \text{ for all } \beta \in B\}.$$

WQO: Permutations $\longleftrightarrow$ graphs

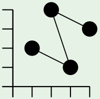
$$\sigma \leq \pi \implies G_\sigma \leq_{\text{ind}} G_\pi$$

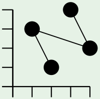
This means

$$\text{Av}(B) \text{ is wqo } \implies \{G_\beta : \beta \in B\}\text{-free permutation graphs are wqo.}$$

Conversely, the perm $\rightarrow$ graph mapping is not injective:

P_4 in two ways





Open Problem

Av(B) is wqo $\overset{?}{\iff}$ $\{G_\beta : \beta \in B\}$-free permutation graphs are wqo.

How to convert antichains

- For a graph G, define

$$\Pi(G) = \{\text{permutations } \pi : G_\pi \cong G\}.$$

  e.g. $\Pi(P_4) = \{2413, 3142\}$, and $\Pi(K_5) = \{54321\}$.

- Given a permutation antichain

$$A = \{\alpha_1, \alpha_2, \dots\},$$

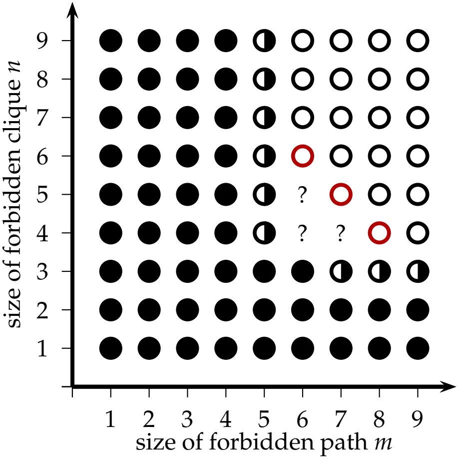
  want each $\Pi(G_{\alpha_i})$, to contain as few permutations as possible.

Fact

$$G_{\alpha_i} \not\leq G_{\alpha_j} \text{ iff } \sigma \not\leq \alpha_j \text{ for all } \sigma \in \Pi(G_{\alpha_i}).$$

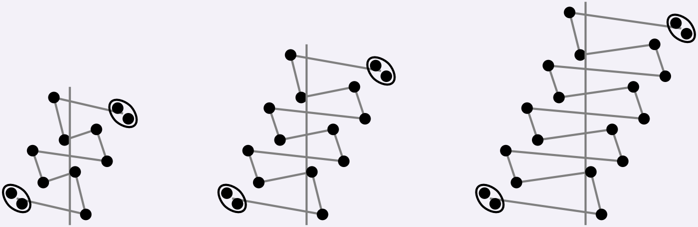
- So for each $\sigma \in \Pi(G_{\alpha_i})$, it suffices to find $\tau \leq \sigma$ such that $\tau \not\leq \alpha_j$ for every j.

Three permutation antichains required



A P_7, K_5-free antichain

An antichain in $\mathrm{Av}(54321, 2416375, 3152746)$ [Murphy, 2003]

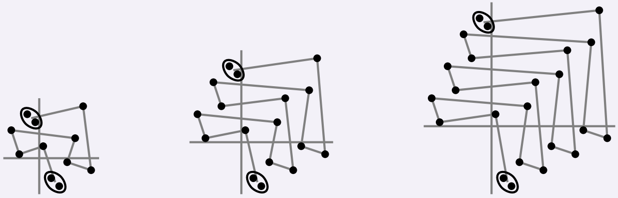


For every π in the above antichain:

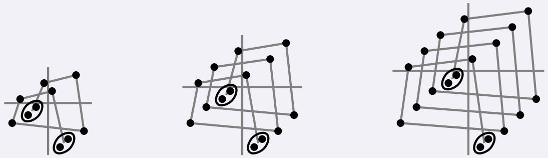
- $|\Pi(G_\pi)| = 4$, and we know what they are.
- $\pi^{-1} \in \Pi(G_\pi)$ contains 51423, but π does not.
- Other permutations in $\Pi(G_\pi)$ can be handled similarly.

The other two antichains

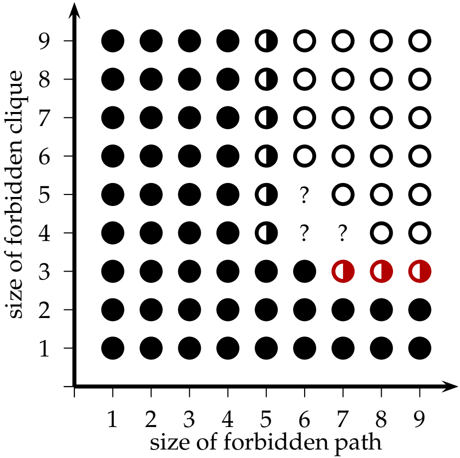
P_6, K_6-free permutation graphs [B., 2012]



P_7, K_4-free permutation graphs [Murphy & Vatter, 2003]

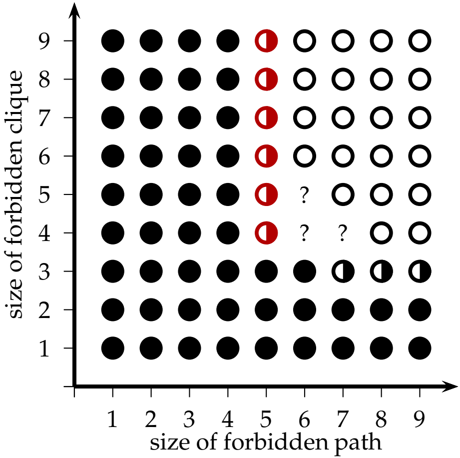


Wqo classes



- Known: P_m, K_3-free permutation graphs are wqo [Lozin and Mayhill, 2011]

Wqo classes

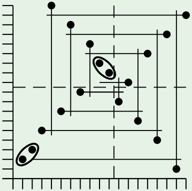


- Known: P_m, K_3-free permutation graphs are wqo [Lozin and Mayhill, 2011]
- Todo: P_5, K_n-free permutation graphs are wqo, for all n.

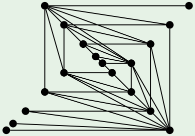
Notes

- P_5, $K_{126923785921975}$-free permutation graphs are wqo, but P_5-free permutation graphs are not wqo.

Here's an antichain element



$\longrightarrow$



- This antichain needs arbitrarily large cliques.

The permutation problem

Theorem

The class of permutations $Av(n \cdots 21, 24153, 31524)$ *is wqo.*

- $G_{n\cdots 21} \cong K_n$
- $G_{24153} \cong G_{31524} \cong P_5$ (and these are the only two permutations).
- So $\mathrm{Av}(n \cdots 21, 24153, 31524)$ corresponds to P_5, K_n-free permutation graphs.

Corollary

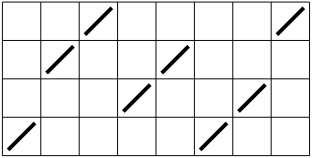
The class of P_5, K_n-free permutation graphs is wqo.

Proving the theorem – Step 1

Proposition

The simple permutations of $Av(n\cdots 21, 24153, 31524)$ are griddable.

- Simple permutations are 'building blocks' (c.f. prime graphs)
- Griddable = can draw on a picture like this:



Proof

- Induction on n.
- Key step: in graph terms, limit the size of the largest matching in a prime graph

What's gridding good for?

Theorem (Albert, Ruškuc, Vatter, 2014)

If the simple permutations in a class are geometrically griddable, then the class is wqo.

'Geometrically griddable' is stricter than 'griddable'

$$\mathrm{GGrid}\left(\begin{array}{|c|c|}\hline \diagdown & \diagup \\ \hline \diagup & \diagdown \\ \hline\end{array}\right) \to P_4\text{-free split permutation graphs}$$

is a subclass of:

$$\mathrm{Grid}\left(\begin{array}{|c|c|}\hline \diagdown & \diagup \\ \hline \diagup & \diagdown \\ \hline\end{array}\right) \to \text{ split permutation graphs}$$

- Aim: take gridding from Step 1 and refine to a geometric one

Step 2 – refine the gridding

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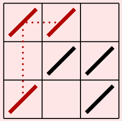
Proposition

The simple permutations of $Av(n \cdots 21, 24153, 31524)$ *are griddable without NW corners.*

NW corners and cycles



- NW corner = configuration shown in red

Step 2 – refine the gridding

Proposition

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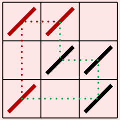
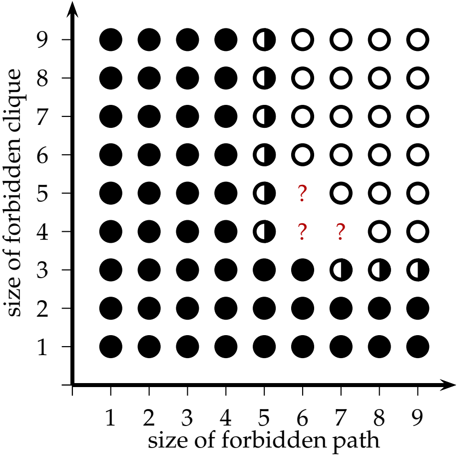


- NW corner = configuration shown in red
- Cycle = closed dotted line
- No NW corners ⇒ no cycles!
- No cycles ⇒ gridding is geometric ⇒ class is wqo

The question marks



- Three classes remain: $\{P_6, K_5\}$, $\{P_6, K_4\}$ and $\{P_7, K_4\}$.
- Not griddable (in the sense used here)
- None of our antichain construction tricks work

Thanks!

Main reference:

Atminas, B., Korpelainen, Lozin & Vatter, *Well-quasi-order for permutation graphs omitting a path and a clique*, arXiv 1312:5907