Unbounded clique-width in hereditary graph classes

Robert Brignall

Based on joint work with Dan Cocks

Queen Mary, University of London, 9th February 2024

The Open University

Grid theorems

Grid minor theorem (Robertson & Seymour, 1986)

A minor-closed class of graphs has bounded tree-width if and only if it excludes a planar graph.

Graph minor: delete vertices or edges, and contract edges.
Tree-width: measures how much a graph is like a tree.

Grid theorems

Grid minor theorem (Robertson & Seymour, 1986)

A minor-closed class of graphs has bounded tree-width if and only if it excludes a planar graph.

Graph minor: delete vertices or edges, and contract edges.
Tree-width: measures how much a graph is like a tree.

Grid theorem for vertex minors (Geelen, Kwon, Mccarty, Wollan, 2023)

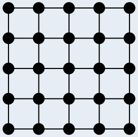
A vertex-minor-closed class of graphs has bounded rank-width if and only if it excludes a circle graph.

Vertex-minor: delete vertices and take ‘local complements’.
Rank-width: a graph measure involving ranks of matrices in certain decompositions of a graph.

Grid theorems – alternative statements

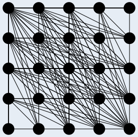
Grid minor theorem (Robertson & Seymour, 1986)

Graphs of large tree-width contain a large grid as a minor.



Grid theorem for vertex minors (Geelen, Kwon, Mccarty, Wollan, 2023)

Graphs of large rank-width contain a large comparability grid as a vertex-minor.



Metatheorems

Theorem (Courcelle, 1990)

Any problem expressible in MSO_2 logic can be solved in linear time on every class of graphs with bounded tree-width.

MSO_2 logic covers problems like existence of perfect matchings, or Hamiltonian cycles.

Metatheorems

Theorem (Courcelle, 1990)

Any problem expressible in MSO_2 logic can be solved in linear time on every class of graphs with bounded tree-width.

MSO_2 logic covers problems like existence of perfect matchings, or Hamiltonian cycles.

Theorem (Courcelle, Makowsky, Rotics, 2000)

Any problem expressible in MSO_1 logic can be solved in linear time on every class of graphs with bounded rank-width.

MSO_1 logic: Weaker than MSO_2, but includes finding a maximum independent set, and deciding k-colourability.

In simple terms

If a collection of graphs has…

…some planar graph as a forbidden minor, lots of graph problems are easy to solve.

In simple terms

If a collection of graphs has…

…some planar graph as a forbidden minor, lots of graph problems are easy to solve.

…some circle graph as a forbidden vertex-minor, not-so-many-lots of graph problems are easy to solve.

In simple terms

If a collection of graphs has…

…some planar graph as a forbidden minor, lots of graph problems are easy to solve.

…some circle graph as a forbidden vertex-minor, not-so-many-lots of graph problems are easy to solve.

Why use anything other than treewidth?

$$\mathrm{tw}(K_n) = n - 1.$$

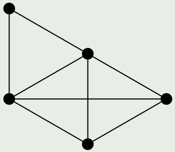
Classes with bounded tree-width can't contain dense graphs.

Is there a ‘grid theorem’ for bounding clique-width in hereditary classes?

Induced subgraphs

- Graph $G = (V, E)$, undirected, simple (no loops, or multiple edges).
- Induced subgraph: $H \leqslant_{\text{ind}} G$ if we can delete vertices (and incident edges) from G to form a graph isomorphic to H.

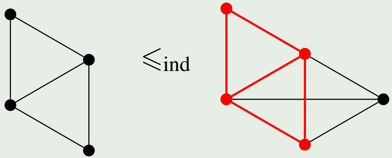
Example (Graphs and induced subgraphs)



Induced subgraphs

- Graph $G = (V, E)$, undirected, simple (no loops, or multiple edges).
- Induced subgraph: $H \leqslant_{\text{ind}} G$ if we can delete vertices (and incident edges) from G to form a graph isomorphic to H.

Example (Graphs and induced subgraphs)



Hereditary classes

Set $\mathcal{C}$ of graphs is hereditary if

$$G \in \mathcal{C} \text{ and } H \leqslant_{\text{ind}} G \text{ implies } H \in \mathcal{C}.$$

'class'

'Closed under induced subgraphs'.

Examples

Forests

Bipartite graphs

Planar graphs

Circle graphs

Permutation graphs

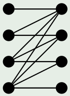
Build-a-graph

You have 4 operations to build a graph:

1. Create a new vertex with a label i.
2. Disjoint union of two previously-constructed graphs.
3. Join all vertices labelled i to all labelled j ($i \neq j$).
4. Relabel every vertex labelled i with j.

Example

Target:

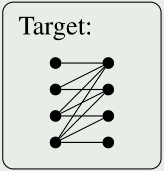


Build-a-graph

You have 4 operations to build a graph:

1. Create a new vertex with a label i.
2. Disjoint union of two previously-constructed graphs.
3. Join all vertices labelled i to all labelled j ($i \neq j$).
4. Relabel every vertex labelled i with j.

Example



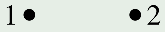
Create vertex with label 1

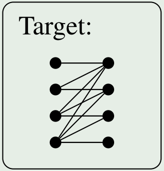
Build-a-graph

You have 4 operations to build a graph:

1. Create a new vertex with a label i.
2. Disjoint union of two previously-constructed graphs.
3. Join all vertices labelled i to all labelled j ($i \neq j$).
4. Relabel every vertex labelled i with j.

Example





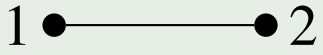
Create vertex with label 2

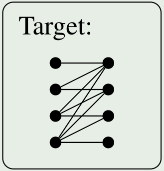
Build-a-graph

You have 4 operations to build a graph:

1. Create a new vertex with a label i.
2. Disjoint union of two previously-constructed graphs.
3. Join all vertices labelled i to all labelled j ($i \neq j$).
4. Relabel every vertex labelled i with j.

Example





Join labels 1 and 2

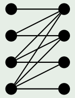
Build-a-graph

You have 4 operations to build a graph:

1. Create a new vertex with a label i.
2. Disjoint union of two previously-constructed graphs.
3. Join all vertices labelled i to all labelled j ($i \neq j$).
4. Relabel every vertex labelled i with j.

Example

Target:



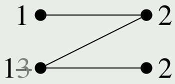
Create vertices with labels 2 and 3 (or use disjoint union)

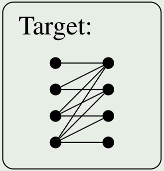
Build-a-graph

You have 4 operations to build a graph:

1. Create a new vertex with a label i.
2. Disjoint union of two previously-constructed graphs.
3. Join all vertices labelled i to all labelled j ($i \neq j$).
4. Relabel every vertex labelled i with j.

Example





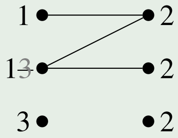
Join labels 2 and 3, and relabel $3 \to 1$

Build-a-graph

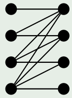
You have 4 operations to build a graph:

1. Create a new vertex with a label i.
2. Disjoint union of two previously-constructed graphs.
3. Join all vertices labelled i to all labelled j ($i \neq j$).
4. Relabel every vertex labelled i with j.

Example



Target:



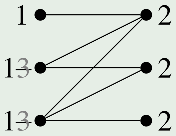
Create vertices with labels 2 and 3 (or use disjoint union)

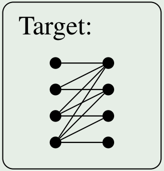
Build-a-graph

You have 4 operations to build a graph:

1. Create a new vertex with a label i.
2. Disjoint union of two previously-constructed graphs.
3. Join all vertices labelled i to all labelled j ($i \neq j$).
4. Relabel every vertex labelled i with j.

Example





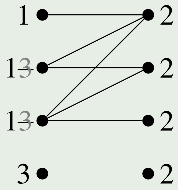
Join labels 2 and 3, and relabel $3 \to 1$

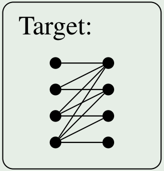
Build-a-graph

You have 4 operations to build a graph:

1. Create a new vertex with a label i.
2. Disjoint union of two previously-constructed graphs.
3. Join all vertices labelled i to all labelled j ($i \neq j$).
4. Relabel every vertex labelled i with j.

Example





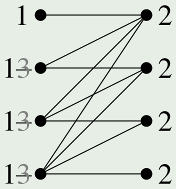
Create vertices with labels 2 and 3 (or use disjoint union)

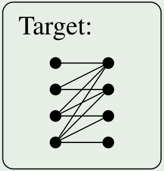
Build-a-graph

You have 4 operations to build a graph:

1. Create a new vertex with a label i.
2. Disjoint union of two previously-constructed graphs.
3. Join all vertices labelled i to all labelled j ($i \neq j$).
4. Relabel every vertex labelled i with j.

Example





Join labels 2 and 3, and relabel $3 \rightarrow 1$

Build-a-graph

You have 4 operations to build a graph:

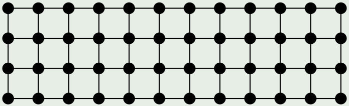
1. Create a new vertex with a label i.
2. Disjoint union of two previously-constructed graphs.
3. Join all vertices labelled i to all labelled j ($i \neq j$).
4. Relabel every vertex labelled i with j.

- Clique-width, $cw(G) =$ size of smallest label set needed to build G.
- If $H \leqslant_{\text{ind}} G$, then $cw(H) \leqslant cw(G)$.
- Clique-width of a class $\mathcal{C}$

$$cw(\mathcal{C}) = \max_{G \in \mathcal{C}} cw(G)$$

if this exists.

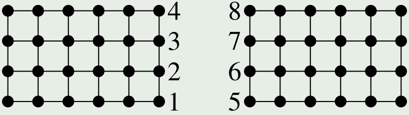
What has big clique width?



For fixed k: $cw(k \times n \text{ grid}) = O(k)$

Intuition: Unbounded clique width needs two dimensions.

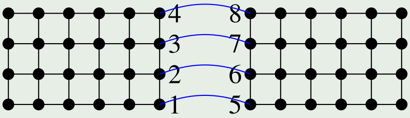
What has big clique width?



For fixed k: $cw(k \times n \text{ grid}) = O(k)$

Intuition: Unbounded clique width needs two dimensions.

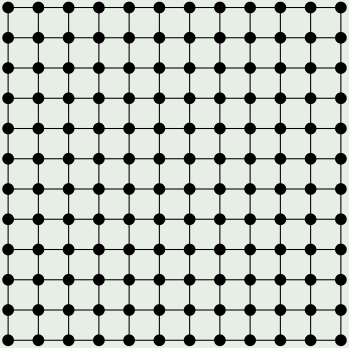
What has big clique width?



For fixed k: $cw(k \times n \text{ grid}) = O(k)$

Intuition: Unbounded clique width needs two dimensions.

What has big clique width?



For fixed k: $cw(k \times n \text{ grid}) = O(k)$

$cw(n \times n \text{ grid}) = n + 1$ (Golumbic and Rotics, 1999)

Intuition: Unbounded clique width needs two dimensions.

Tree-width, rank-width, clique-width

Theorem (Corneil and Rotics, 2005)

For any graph G,

$$cw(G) \leqslant 3 \cdot 2^{tw(G)}.$$

Note: $cw(K_n) = 2$, but $tw(K_n) = n - 1$.

Theorem (Oum and Seymour, 2006)

For any graph G,

$$rw(G) \leqslant cw(G) \leqslant 2^{rw(G)+1} - 1.$$

Thus:

- Clique-width unbounded implies tree-width unbounded (converse false)
- Rank-width unbounded iff clique-width unbounded

Usefulness of clique-width

Since rank-width and clique-width are related:

Theorem (Courcelle, Makowsky, Rotics, 2000)

Any problem expressible in MSO_1 logic can be solved in linear time on every class of graphs with bounded rank-width.

Usefulness of clique-width

Since rank-width and clique-width are related:

Theorem (Courcelle, Makowsky, Rotics, 2000)

Any problem expressible in MSO_1 logic can be solved in linear time on every class of graphs with bounded clique width.

(In fact, rank-width was only introduced in 2006, so this is more like the original result.)

Is there a ‘grid theorem’ for bounding clique-width in hereditary classes?

Just use the vertex-minor grid theorem?

Hereditary classes are a richer (and arguably more natural) family:
every vertex-minor-closed class is hereditary.

The ‘circle graphs’ in the vertex-minor grid theorem contain lots of interesting hereditary classes. Some have bounded clique-width, others don’t.

Clique-width: history to 2011

1993 Courcelle, Engelfriet & Rozenberg: (sort of) introduce clique-width.

1999 Makowsky & Rotics: *split graphs* have unbounded clique-width.

2000 Courcelle, Makowsky & Rotics: MSO_1 metatheorem.

Golumbic & Rotics: *permutation graphs* have unbounded clique-width.

2006 Oum & Seymour introduce rank-width as an approximation for clique-width that can be computed efficiently.

2011 Lozin shows that bipartite permutation graphs and unit interval graphs are minimal classes with unbounded clique-width.

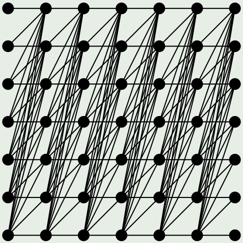
Minimal hereditary classes of unbounded clique-width

Class $\mathcal{C}$ is *minimal (of unbounded clique-width)* if:

- $\mathcal{C}$ has unbounded clique-width, and
- any proper subclass $\mathcal{D} \subsetneq \mathcal{C}$ has bounded clique-width.

Bipartite permutation graphs (Lozin, 2011)

Class comprises all induced subgraphs of grids like the following:



Is there a ‘grid theorem’ for bounding clique-width in hereditary classes?

Is there a ‘grid theorem’ for bounding clique-width in hereditary classes?

Not if you want it to mention just one ‘grid’

Since bipartite permutation graphs and unit interval graphs are both minimal of unbounded clique-width, there are at least two grids. . .

Is there a 'grid theorem' for bounding clique-width in hereditary classes?

Not if you want it to mention just one 'grid'

Since bipartite permutation graphs and unit interval graphs are both minimal of unbounded clique-width, there are at least two grids...

...but perhaps we could list the minimal classes?

Discovering minimal classes

2011 Lozin: bipartite permutation graphs and unit interval graphs.

Discovering minimal classes

2011 Lozin: bipartite permutation graphs and unit interval graphs.

2015 (published 2021) Atminas, B., Lozin & Stacho:
split permutation graphs and bichain graphs.

Discovering minimal classes

2011 Lozin: bipartite permutation graphs and unit interval graphs.

2015 (published 2021) Atminas, B., Lozin & Stacho:
split permutation graphs and bichain graphs.

2018 Collins, Foniok, Korpelainen, Lozin & Zamaraev:
countably infinite collection of minimal classes

Discovering minimal classes

2011 Lozin: bipartite permutation graphs and unit interval graphs.

2015 (published 2021) Atminas, B., Lozin & Stacho:
split permutation graphs and bichain graphs.

2018 Collins, Foniok, Korpelainen, Lozin & Zamaraev:
countably infinite collection of minimal classes

2022 B. & Cocks: uncountably infinite collection

Discovering minimal classes

2011 Lozin: bipartite permutation graphs and unit interval graphs.

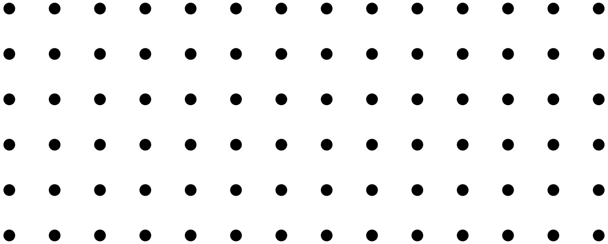
2015 (published 2021) Atminas, B., Lozin & Stacho:
split permutation graphs and bichain graphs.

2018 Collins, Foniok, Korpelainen, Lozin & Zamaraev:
countably infinite collection of minimal classes

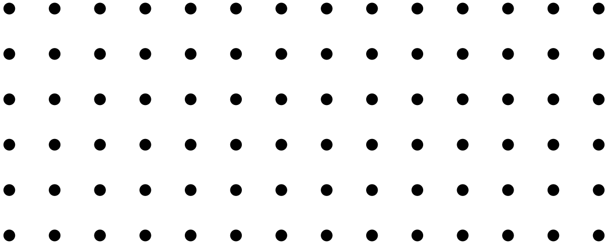
2022 B. & Cocks: uncountably infinite collection

2023 B. & Cocks: General framework for all the above classes.

The framework



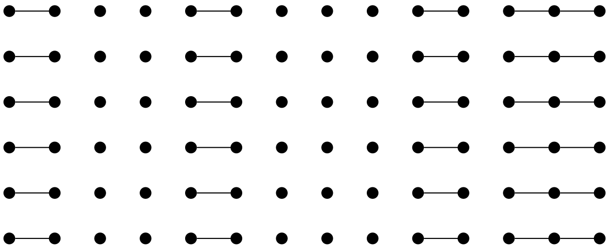
The framework



$\alpha \quad 0 \; 1 \; 2 \; 1 \; 0 \; 3 \; 3 \; 2 \; 2 \; 0 \; 2 \; 0 \; 0 \cdots \quad \in \{0, 1, 2, 3\}^*$

In α: $0 =$ matching, $1 =$ co-matching, $2 =$ half-graph, $3 =$ co-half-graph.

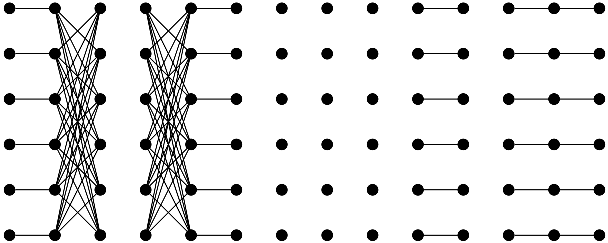
The framework



α $\quad 0 \quad 1 \quad 2 \quad 1 \quad 0 \quad 3 \quad 3 \quad 2 \quad 2 \quad 0 \quad 2 \quad 0 \quad 0 \cdots \quad \in \{0, 1, 2, 3\}^*$

In α: $0 =$ matching, $1 =$ co-matching, $2 =$ half-graph, $3 =$ co-half-graph.

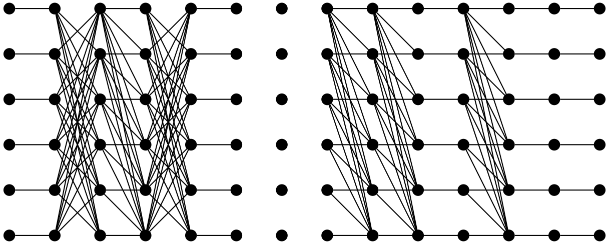
The framework



α 0 1 2 1 0 3 3 2 2 0 2 0 0 $\cdots$ $\in \{0, 1, 2, 3\}^*$

In α: $0 =$ matching, $1 =$ co-matching, $2 =$ half-graph, $3 =$ co-half-graph.

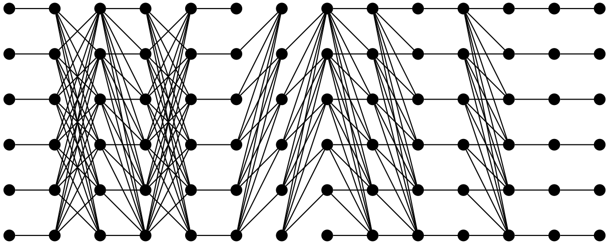
The framework



α 0 1 2 1 0 3 3 2 2 0 2 0 0 $\cdots$ $\in \{0, 1, 2, 3\}^*$

In α: 0 = matching, 1 = co-matching, 2 = half-graph, 3 = co-half-graph.

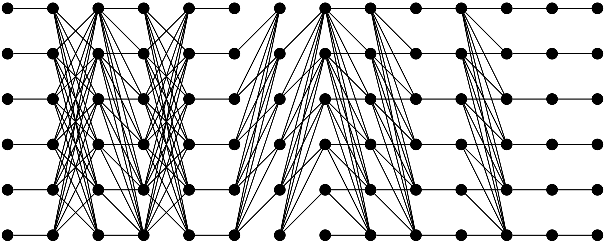
The framework



$\alpha \quad 0 \quad 1 \quad 2 \quad 1 \quad 0 \quad 3 \quad 3 \quad 2 \quad 2 \quad 0 \quad 2 \quad 0 \quad 0 \quad \cdots \qquad \in \{0, 1, 2, 3\}^*$

In α: $0 =$ matching, $1 =$ co-matching, $2 =$ half-graph, $3 =$ co-half-graph.

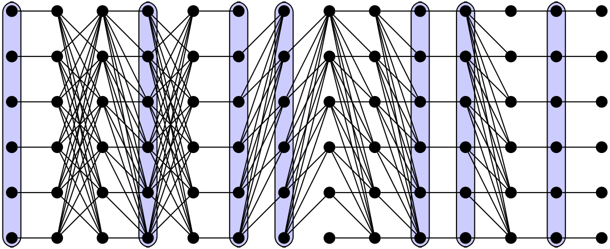
The framework



$\beta \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad \cdots \quad \in \{0,1\}^*$

In β: $0 =$ independent set, $1 =$ clique.

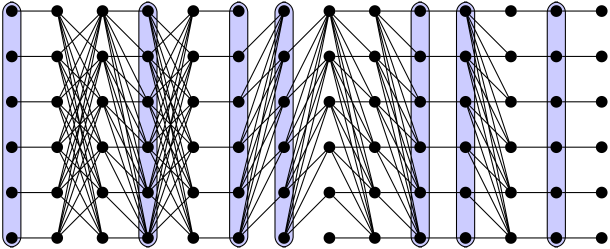
The framework



β 1 0 0 1 0 1 1 0 0 1 1 0 1 0 $\cdots$ $\in \{0,1\}^*$

In β: 0 = independent set, 1 = clique.

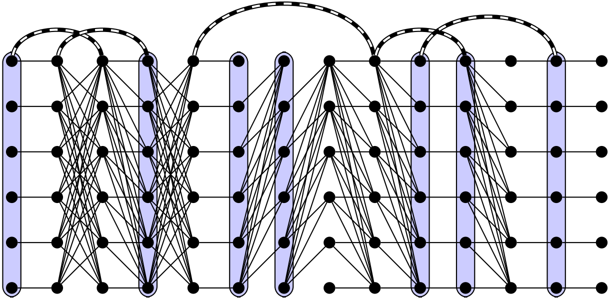
The framework



γ $\quad (1,3),(2,4),(5,9),(9,11),(10,13),\ldots \qquad \subseteq \mathbb{N}\times\mathbb{N}$

In γ: (i,j) joins all of column i to column j.

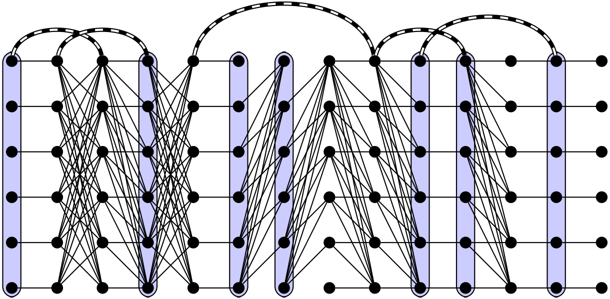
The framework



$\gamma \qquad (1,3),(2,4),(5,9),(9,11),(10,13),\ldots \qquad \subseteq \mathbb{N}\times\mathbb{N}$

In γ: (i,j) joins all of column i to column j.

The framework



α 0 1 2 1 0 3 3 2 2 0 2 0 0 $\cdots$ $\in \{0,1,2,3\}^*$

β 1 0 0 1 0 1 1 0 0 1 1 0 1 0 $\cdots$ $\in \{0,1\}^*$

γ $(1,3),(2,4),(5,9),(9,11),(10,13),\ldots$ $\subseteq \mathbb{N}\times\mathbb{N}$

In α: $0 =$ matching, $1 =$ co-matching, $2 =$ half-graph, $3 =$ co-half-graph.

In β: $0 =$ independent set, $1 =$ clique.

In γ: (i,j) joins all of column i to column j.

From grids to classes

Each triple $\delta = (\alpha, \beta, \gamma)$ defines an infinite graph H^δ, and then

$$\mathcal{G}^\delta = \{G \text{ finite} : G \leqslant_{\text{ind}} H^\delta\}.$$

From grids to classes

Each triple $\delta = (\alpha, \beta, \gamma)$ defines an infinite graph H^δ, and then

$$\mathcal{G}^\delta = \{G \text{ finite} : G \leqslant_{\text{ind}} H^\delta\}.$$

Theorem (B. & Cocks, 2023)

There exists a parameter N^δ relying only on δ such that $cw(\mathcal{G}^\delta)$ is unbounded if and only if N^δ is unbounded.

N^δ is unbounded, for example, if α contains infinitely many 2s or 3s.

Uncountably many minimal classes

Not all classes $\mathcal{G}^{\delta}$ of unbounded clique width are minimal, but lots are. One simple family is as follows:

Theorem (B. & Cocks, 2022)

Let $\beta = 00\cdots$, $\gamma = \varnothing$ and let α be any infinite uniformly recurrent word over the alphabet $\{0, 1, 2, 3\}$ other than $00\cdots$.
Then $\mathcal{G}^{(\alpha,\beta,\gamma)}$ is a minimal hereditary class of unbounded clique width.

Uniformly recurrent: every factor w of α is guaranteed to appear in α infinitely often, and consecutive occurrences are ‘close’.

Uncountably many minimal classes

Not all classes $\mathcal{G}^{\delta}$ of unbounded clique width are minimal, but lots are. One simple family is as follows:

Theorem (B. & Cocks, 2022)

Let $\beta = 00\cdots$, $\gamma = \varnothing$ and let α be any infinite uniformly recurrent word over the alphabet $\{0, 1, 2, 3\}$ other than $00\cdots$.
Then $\mathcal{G}^{(\alpha,\beta,\gamma)}$ is a minimal hereditary class of unbounded clique width.

Uniformly recurrent: every factor w of α is guaranteed to appear in α infinitely often, and consecutive occurrences are 'close'.

Sturmian sequences are an uncountably large collection of uniformly recurrent binary sequences $\Rightarrow$ uncountably many minimal classes.

Is there a ‘grid theorem’ for bounding clique-width in hereditary classes?

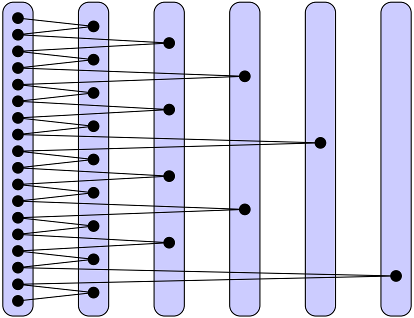
Is there a ‘grid theorem’ for bounding clique-width in hereditary classes?

Well, you can’t list them all…

…but perhaps we are now close to a complete characterisation?

Dragons

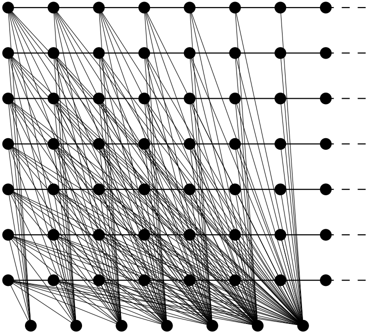
‘Power graphs’ discovered by Lozin, Razgon & Zamaraev (2018), proved minimal by Dawar & Sankaran (2023):



Not part of the previous framework. Also, this class is well-quasi-ordered (if you know what that means).

Worse Dragons

Korpelainen (2016): The following class has unbounded clique width but does not contain a minimal class:

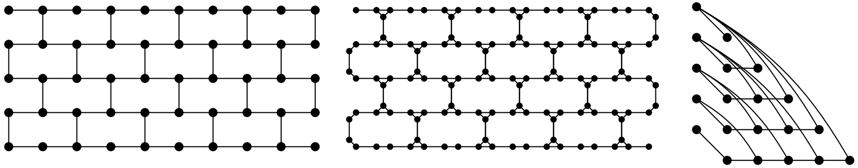


Sparse dragons

Theorem (Gurski & Wanke, 2000)

If a hereditary class is sparse, then it has unbounded clique-width if and only if it has unbounded tree-width.

Classifying bounded tree-width in sparse graphs is a major topic in itself.

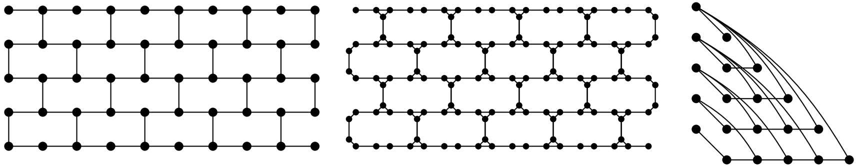


Sparse dragons

Theorem (Gurski & Wanke, 2000)

If a hereditary class is sparse, then it has unbounded clique-width if and only if it has unbounded tree-width.

Classifying bounded tree-width in sparse graphs is a major topic in itself.



Conjecture (Cocks, 2024+)

Sparse hereditary graph classes of unbounded tree-width do not contain a minimal class of unbounded tree-width.

Thanks!

Main references:

- B. & Cocks, *Uncountably many minimal hereditary classes of graphs of unbounded clique-width*, Elec. J. Combin. **29** (2022)
- B. & Cocks, *A framework for minimal hereditary classes of graphs of unbounded clique-width*, SIAM J. Disc. Math. **37** (2023)