Simple Permutations

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Outline

Permutations

- Permutation of length n: an ordering on the symbols $1, \ldots, n$.
- For example: $\pi = 15482763$.

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- Graphical viewpoint: plot the points $(i, \pi(i))$.

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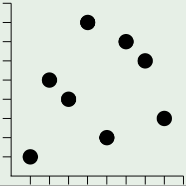

Intervals

- Pick any permutation π.
- An interval of π is a set of contiguous indices $I = [a, b]$ such that $\pi(I) = \{\pi(i) : i \in I\}$ is also contiguous.

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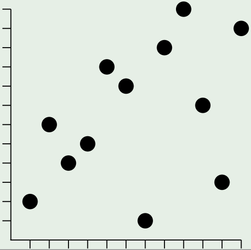

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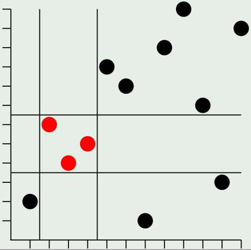

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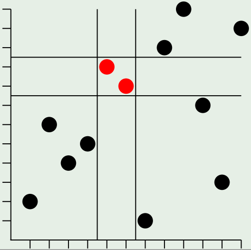

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- Intervals are important in biomathematics (genetic algorithms, matching gene sequences).

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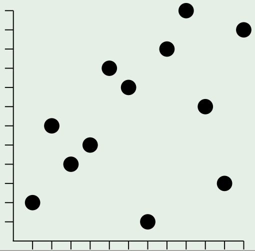

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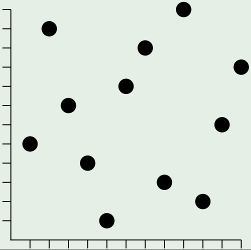

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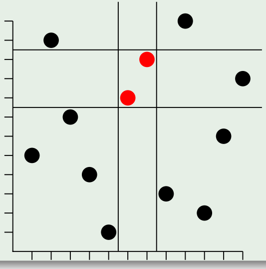

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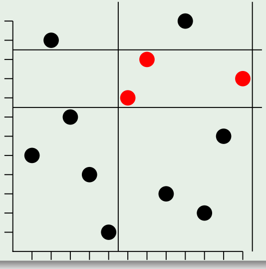

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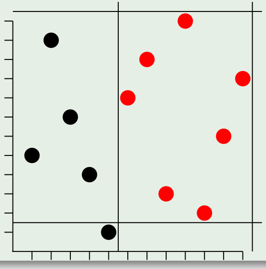

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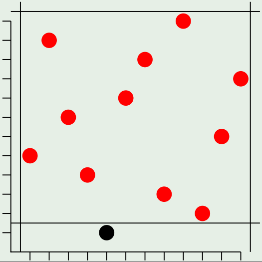

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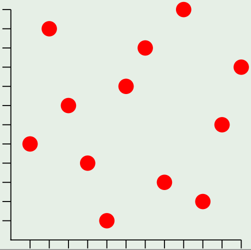

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Theorem (Albert, Atkinson and Klazar, 2003)

The number of simple permutations is asymptotically given by

$$\frac{n!}{e^2}\left(1 - \frac{4}{n} + \frac{2}{n(n-1)} + O(n^{-3})\right).$$

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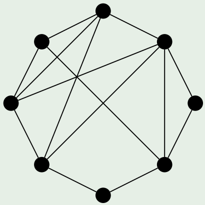
- Binary relations come in many different flavours – linear, transitive, symmetric,...
- Relational structures include graphs, digraphs, tournaments and posets.

Graphs

- Graph — a relational structure on a single binary symmetric relation.

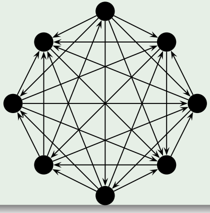
Example



Tournaments

- Tournament — a complete oriented graph.
- Formed by a single trichotomous binary relation — $x \to y$, $y \to x$ or $x = y$.

Example



Two Linear Orders

- Permutation of length n — a structure on two linear relations.

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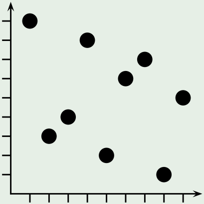

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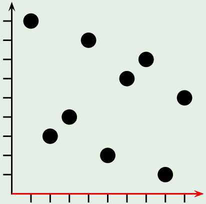

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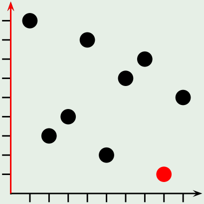

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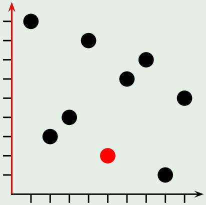

- $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9.$
- $8 \prec 5$

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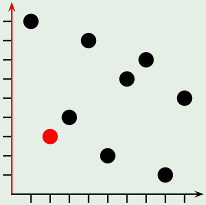

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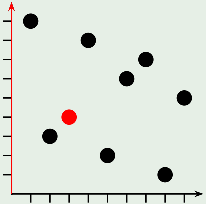

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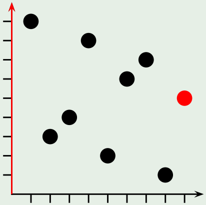

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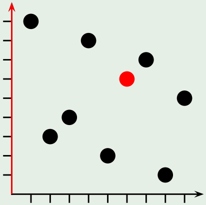

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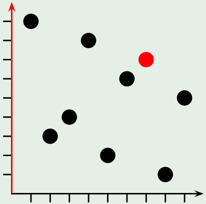

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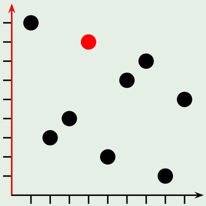

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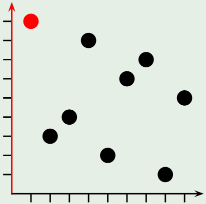

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- Graphs — indecomposable or prime graphs.

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- Tournaments may be written as an abstract algebra with two idempotent binary operations, $\vee$ and $\wedge$.
- If $x \rightarrow y$ in the tournament, then $x \vee y = x$ and $x \wedge y = y$.
- Simple tournament $\iff$ simple abstract algebra.
- (The kernel of every homomorphism is either the whole structure or a single element.)

The Substitution Decomposition

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- Frequently rediscovered in different settings under various names: modular decomposition, disjunctive decomposition, X-join...
- Möhring (1985), and Möhring and Radermacher (1984) discuss applications in combinatorial optimisation and game theory.

Decomposing Permutations

- Break permutation into maximal proper intervals.

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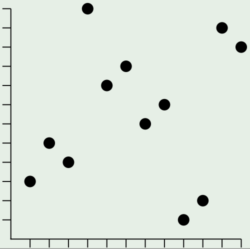

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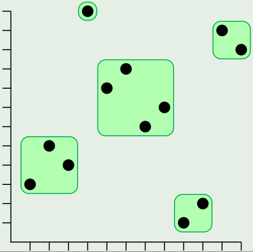

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- Gives a unique simple permutation, the skeleton.

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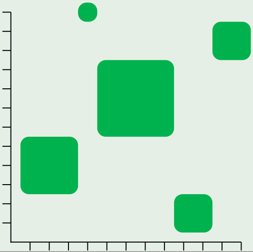

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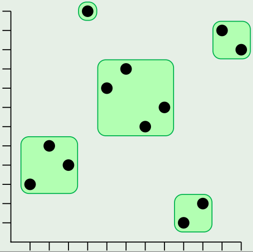

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- This decomposition is the substitution decomposition.

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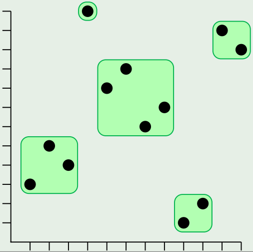

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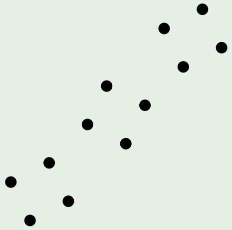

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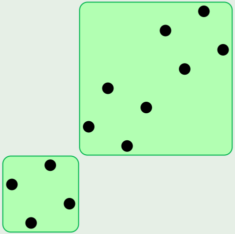

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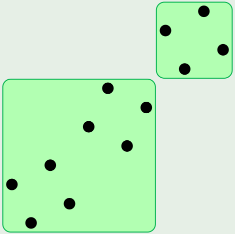

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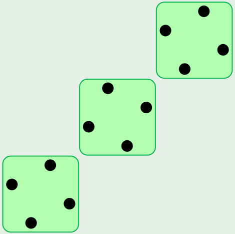

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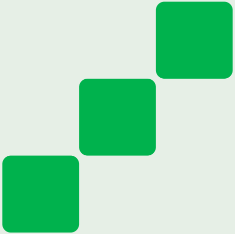

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Pattern Containment

- A permutation $\tau = t_1 t_2 \ldots t_k$ is contained in the permutation $\sigma = s_1 s_2 \ldots s_n$ if there exists a subsequence $s_{i_1}, s_{i_2}, \ldots, s_{i_k}$ order isomorphic to τ.

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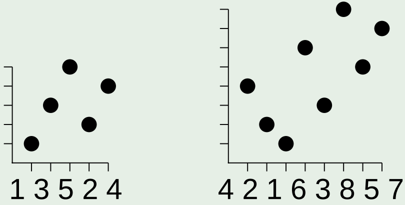

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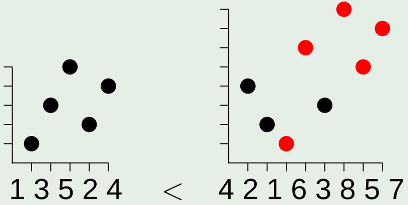

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- Viewing permutations as relational structures, containment corresponds to:
  - taking subsets of the ground set $A = [n]$,
  - restricting the two linear orders to act only on the subset.
- Easily generalise this to all relational structures.
- For example, in graphs, containment corresponds to taking induced subgraphs.

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- Pattern containment is easily restricted to the containment of simple permutations within other simple permutations.
- Get another partial order on the set of all simple permutations. What does it look like?
- In fact, this theorem is proved for all binary irreflexive relational structures.
- Some generalisations to single k-ary relations made by Ehrenfeucht and McConnell (1994).

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- Which simple permutations of length n contain no simple permutations of length $n - 1$?
- Schmerl and Trotter give criteria for posets and graphs.

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Corollary (Albert and Atkinson, 2005)

The only simple permutations that do not have a one-point simple deletion are those of the form

$$246\cdots(2m)135\cdots(2m-1) \quad (m \geq 2)$$

and every symmetry of this permutation.

Decomposing the Indecomposable

- Erdős and Szekeres (1935): every permutation of length n contains a monotone permutation of length at least $\sqrt{n}$.

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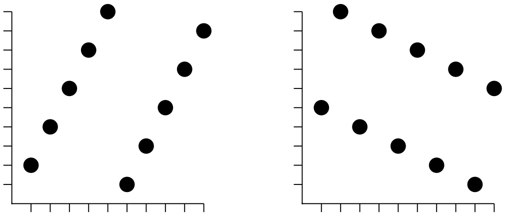
- Erdős and Szekeres (1935): every permutation of length n contains a monotone permutation of length at least $\sqrt{n}$.
- Can we do something similar, restricting our view to simple permutations?
- It would have a number of consequences for “permutation classes”.

Special Simple Permutations



- Parallel alternations (no simple one-point deletion).

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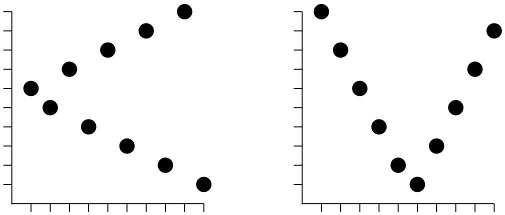

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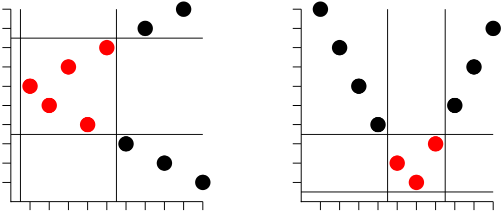
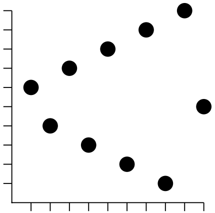


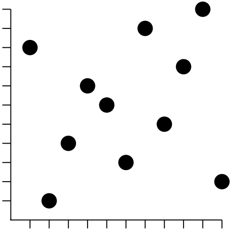
- Wedge alternations – not simple!

Special Simple Permutations



- Two flavours of wedge simple alternation.

Proper Pin Sequences



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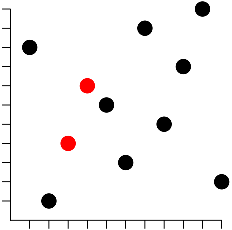

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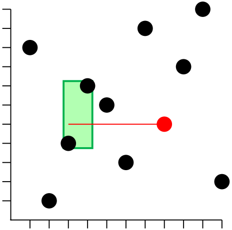

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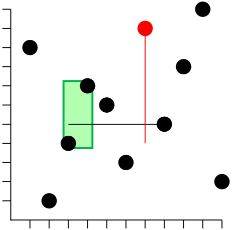

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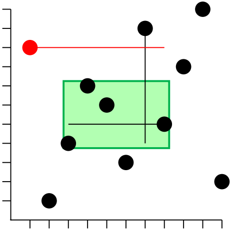

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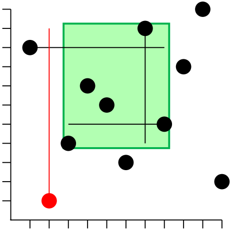

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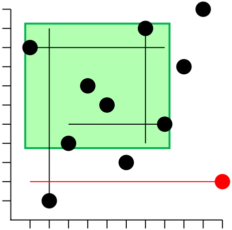

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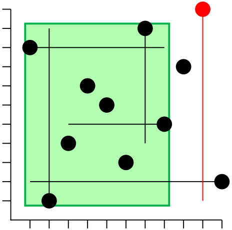

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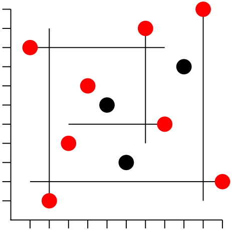

- The points of the proper pin sequence form a simple permutation.

A Decomposition Theorem

Theorem (B., Huczynska and Vatter)

Every simple permutation of length at least $2(256k^8)^{2k}$ contains either a proper pin sequence of length $2k$, a parallel alternation of length $2k$, or a wedge simple permutation of length $2k$.

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- Use Erdős-Szekeres to make parallel or wedge alternations.
- More playing with pin sequences produces wedge simple permutations.

Outline

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Example

The class $\mathcal{C} = \mathrm{Av}(12)$ consists of all the decreasing permutations:

$$\{1, 21, 321, 4321, \ldots\}$$

Permutation Classes II

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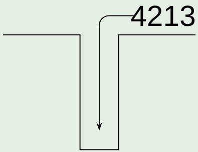

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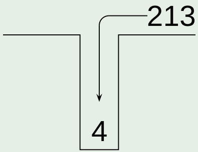

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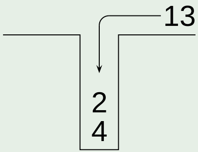

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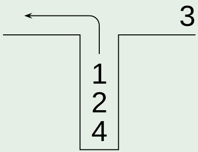

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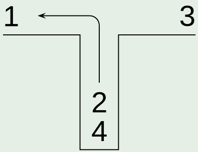

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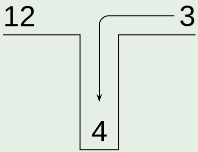

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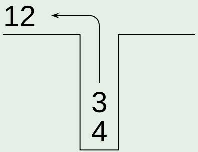

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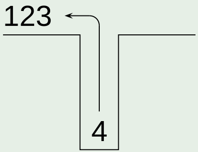

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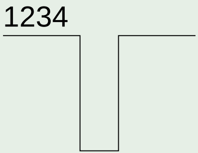

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- We are interested in classes containing only finitely many simple permutations.

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- A finite choice of skeletons $\Rightarrow$ only finite antichains.

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Lemma

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- Smaller classes are subsets of substitution closed classes.
- Bases are antichains, and antichains are finite.

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Question

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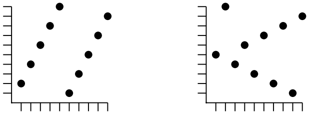
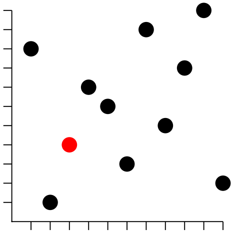


The Language of Pins



- Encode as: 1

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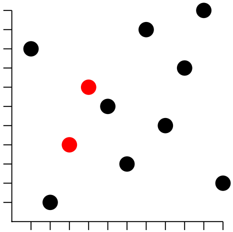

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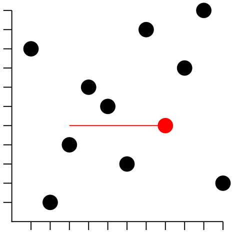

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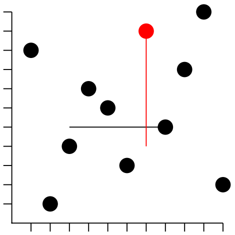

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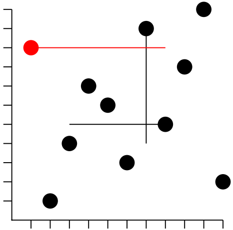

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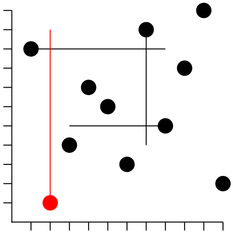

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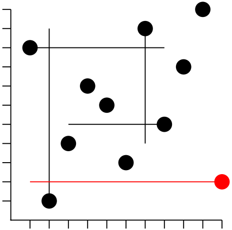

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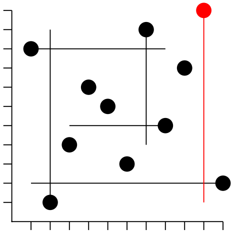

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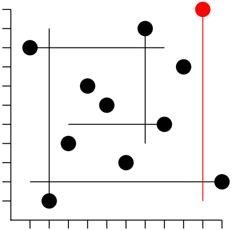

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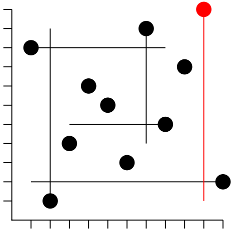

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- Pattern containment $\leftrightarrow$ partial order on pin words.
- Avoiding a pattern $\leftrightarrow$ avoiding every pin word generating that pattern.

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- Parallel and wedge simple permutations easily verified.

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231-avoiders

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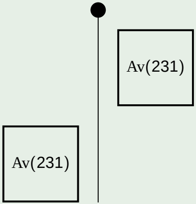

- 231-avoiders: generic structure.

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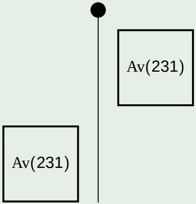

- Only simple permutations are1, 12, and 21.

231-avoiders

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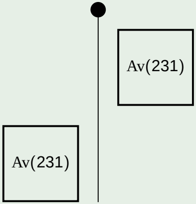

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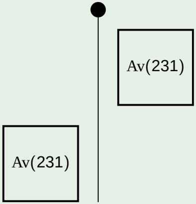

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$$f(x) = \frac{1 - \sqrt{1-4x}}{2x} = 1 + x + 2x^2 + 5x^3 + 14x^4 + \dots$$

Recursion

"...the standard intuition of what a family with an algebraic generating function looks like: the algebraicity suggests that it may (or should...), be possible to give a recursive description of the objects based on disjoint union of sets and concatentation of objects."

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- Recursive description: the substitution decomposition.

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- Permutation classes with only finitely many simple permutations: long permutations are built recursively from much shorter ones.

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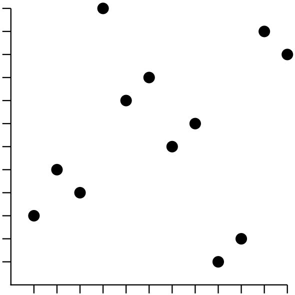
Theorem (Albert and Atkinson, 2005)

A permutation class with only finitely many simple permutations has a readily computable algebraic generating function.

A Spot of Notation



- $\pi = 354C896712BA$.

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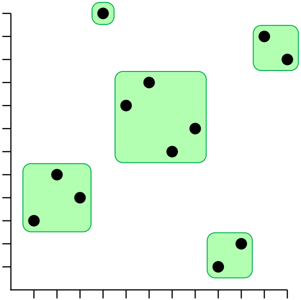

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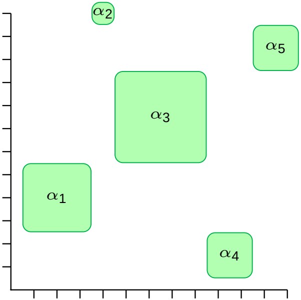

- $\pi = 354C896712BA$.
- $\pi = 25314[132, 1, 3412, 12, 21]$.
- In general: $\pi = \sigma[\alpha_1, \ldots, \alpha_m]$.

Algebraic Generating Functions

Theorem (B., Huczynska and Vatter)

Let $\mathcal{C}$ be a permutation class containing only finitely many simple permutations, $\mathcal{P}$ a finite query-complete set of properties, and $\mathcal{Q} \subseteq \mathcal{P}$. The generating function for the set of permutations in $\mathcal{C}$ satisfying every property in $\mathcal{Q}$ is algebraic over $\mathbb{Q}(x)$.

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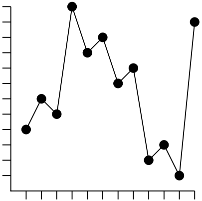
- Finite query-complete: set of query-complete properties is finite.

Some Query-Complete Properties

- The permutations of a class (Albert and Atkinson).
- Alternating permutations.
- Even permutations.
- Dumont permutations.
- Permutations avoiding “blocked” or “barred” patterns.
- Involutions (more work required).
- Any combination of the above.

An Example



- *AL* — Alternating permutations.

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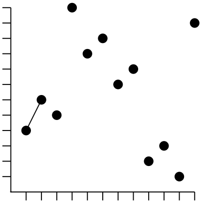

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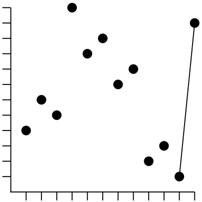

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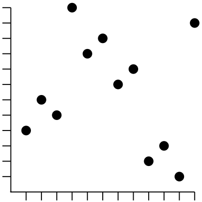

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- Claim: $\{AL, BR, ER, \{1\}\}$ is query-complete.

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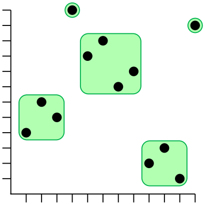

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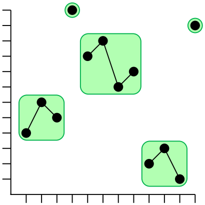

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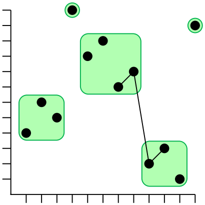

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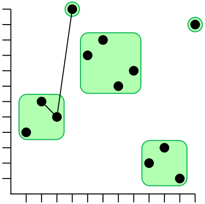

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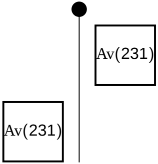
- $\sigma(i) < \sigma(i+1)$: $\alpha_i \notin ER$ and $\alpha_{i+1} \notin BR$.

The Rest of the Details



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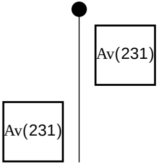

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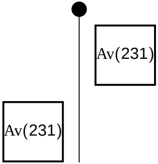

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Theorem (See, e.g., Stanley (1999))

Every proper algebraic system over $\mathbb{Q}[x]$ has a unique solution, which is algebraic over $\mathbb{Q}(x)$.