Characterising structure in classes with unbounded clique-width

Robert Brignall

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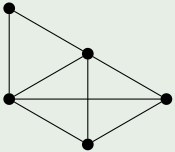
Setup

- Graph $G = (V, E)$, undirected, simple (no loops, or multiple edges).
- Induced subgraph: $H \leq_{\text{ind}} G$.

Example (Graphs and induced subgraphs)



- Class: $\mathcal{C}$, a hereditary property of graphs:

$$G \in \mathcal{C} \text{ and } H \leq_{\text{ind}} G \implies H \in \mathcal{C}.$$

(Example: set of all planar graphs.)

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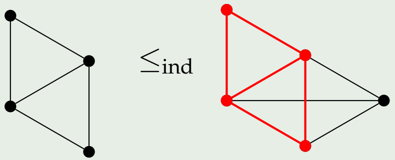

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Build-a-graph

Set of labels Σ. You have 4 operations to build a labelled graph:

1. Create a new vertex with a label $i \in \Sigma$.
2. Disjoint union of two previously-constructed graphs.
3. Join all vertices labelled i to all labelled j, where $i, j \in \Sigma$, $i \neq j$.
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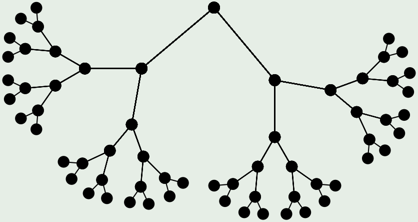
Example (Binary trees need at most 3 labels)



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- Clique-width, $cw(G) =$ size of smallest Σ needed to build G.
- If $H \leq_{\text{ind}} G$, then $cw(H) \leq cw(G)$.
- Clique-width of a class $\mathcal{C}$

$$cw(\mathcal{C}) = \max_{G \in \mathcal{C}} cw(G)$$

if this exists.

Motivation

Theorem (Courcelle, Makowsky and Rotics (2000))

If $cw(\mathcal{C}) < \infty$, then any property expressible in monadic second-order (MSO_1) logic can be determined in polynomial time for $\mathcal{C}$.

- MSO_1 includes many NP-hard algorithms: e.g. k-colouring ($k \geq 3$), graph connectivity, maximum independent set,...
- Generalises treewidth, critical to the proof of the Graph Minor Theorem (see next slide)
- Unlike treewidth, clique-width can cope with dense graphs

Diversion: treewidth, $tw(G)$

- $tw(G)$ measures 'how like a tree' G is ($tw(G) = 1$ iff G is a tree).
- Bounded treewidth $\Longrightarrow$ all problems in MSO_2 in polynomial time.

Theorem (Robertson and Seymour, 1986)

For a minor-closed family of graphs $\mathcal{C}$, $tw(\mathcal{C})$ bounded if and only if $\mathcal{C}$ does not contain all planar graphs.

- Planar graphs are the unique "minimal" family for treewidth.

Question

Can we get a similar theorem for clique width?

Plan for the rest of today

- Bounded vs unbounded clique-width
- Look at minimal classes with unbounded clique-width
- See how permutations can help here
- Compare clique-width with *linear* clique-width
- Look at connections with well-quasi-ordering

Bounding clique-width

Question

Given a class $\mathcal{C}$, is $cw(\mathcal{C})$ bounded?

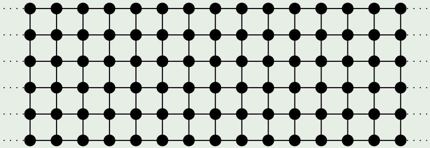
- $cw(G) \leq 3 \cdot 2^{tw(G)-1}$ (Corneil and Rotics, 2005).
- Rank-width: $rw(G) \leq cw(G) \leq 2^{rw(G)+1} - 1$ (Oum and Seymour, 2006) – critical for algorithmic consequences.

Example (Classes of bounded clique-width)

- $\mathcal{F}$ = the class of all forests. $cw(\mathcal{F}) = 3$.
- $\mathcal{C}$ = all cographs
  $= \{G : G \text{ built from } \bullet \text{ by disjoint union and join}\}$
  $cw(\mathcal{C}) = 2$.

What has unbounded clique-width?

Graphs from grids

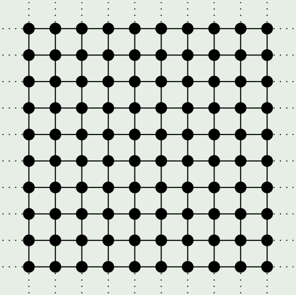


$n \times k$ grids, fixed k: $cw = O(k)$

- Intuition: Unbounded clique width needs two dimensions of complexity.

What has unbounded clique-width?

Graphs from grids



$n \times n$ grids: $cw = n + 1$ (Golumbic and Rotics, 1999)

- Intuition: Unbounded clique width needs two dimensions of complexity.

Classes of unbounded clique-width

Plenty of examples:

- Unit interval graphs (intersection graph of unit-length intervals)
- Split graphs (partition into clique and independent set)
- Bipartite permutation graphs (see later)
- Any class with superfactorial speed
  ($\sim$ more than n^{cn} labelled graphs of order n, for any c)
- Modifications to the $n \times n$ grid gives many more...

Question

Which classes of graphs are minimal with unbounded clique-width?

Minimal classes of unbounded clique-width

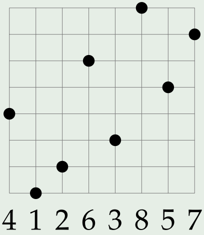
These are rarer (there's more to prove). Four known:

- Unit interval graphs [Lozin, 2011]
- Bipartite permutation graphs [Lozin, 2011]
- Split permutation graphs [Atminas, B., Lozin, Stacho, 2015+]
- Bichain graphs [Atminas, B., Lozin, Stacho, 2015+]

General method to prove minimality of $\mathcal{C}$

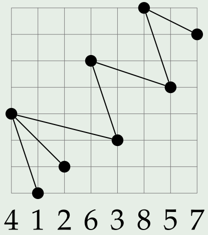
1. Get a structural characterisation of $\mathcal{C}$
2. Find universal graphs U_n: contain all graphs in $\mathcal{C}$ on n vertices
3. Show $cw(U_n) = f(n)$, for some suitably-growing f.
4. Technical lemma: forbidding some $U_n \in \mathcal{C}$ bounds cw.

Permutations and permutation graphs



- Permutation $\pi = \pi(1) \cdots \pi(n)$
- Make a graph G_π: for $i < j$, $ij \in E(G_\pi)$ iff $\pi(i) > \pi(j)$.
- Note: $n \cdots 21$ becomes K_n.

Permutations and permutation graphs



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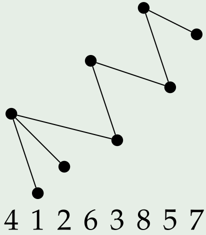
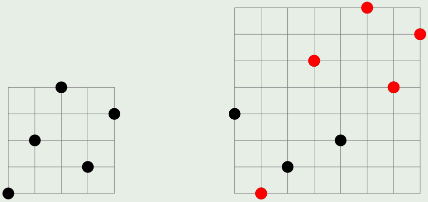


- Permutation graph = can be made from a permutation

Ordering permutations: containment



1 3 5 2 4 $<$ 4 1 2 6 3 8 5 7

- Pattern containment: a partial order, $\sigma \leq \pi$.

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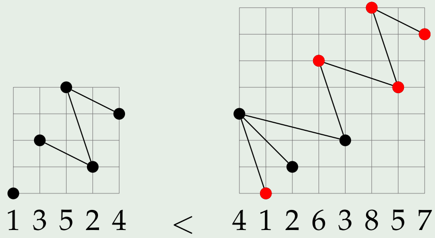

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- Draw the graphs: $G_\sigma \leq_{\text{ind}} G_\pi$.

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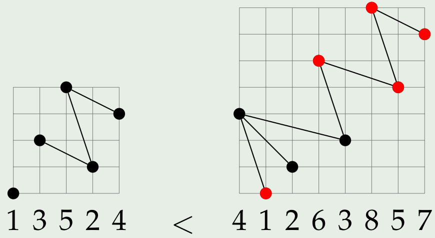

- Pattern containment: a partial order, $\sigma \le \pi$.
- Draw the graphs: $G_\sigma \le_{\text{ind}} G_\pi$.
- Permutation class: hereditary collection

$$\pi \in \mathcal{C} \text{ and } \sigma \le \pi \text{ implies } \sigma \in \mathcal{C}.$$

- Avoidance: minimal forbidden permutation characterisation:

$$\mathcal{C} = \mathrm{Av}(B) = \{\pi : \beta \not\le \pi \text{ for all } \beta \in B\}.$$

Av(321) vs Bipartite permutation graphs

Theorem (Lozin, 2011)

Bipartite permutation graphs are a minimal class with unbounded clique-width.

Permutations	Graphs
$\pi = 321$	$G_\pi =$ 

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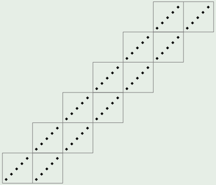
Structure:		

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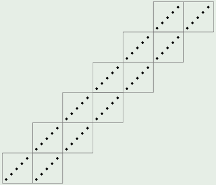
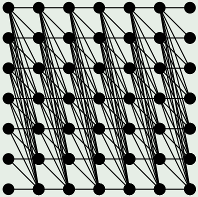
Structure:		

Split permutation graphs

Theorem (Atminas, B., Lozin, Stacho, 2015+)

Split permutation graphs are a minimal class with unbounded clique-width.

Split graph = partition vertices into clique and independent set.

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Merge of $1 \dots k, j \dots 1$	Indep set + clique

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Class:	$\mathrm{Av}(2143, 3412)$	Split permutation

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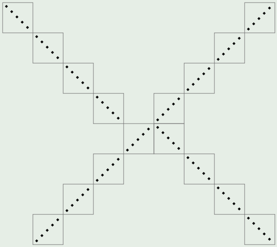
Structure:		

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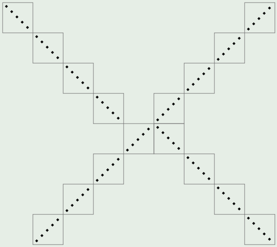
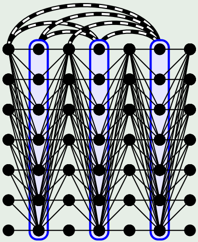
Structure:		

Bichain graphs

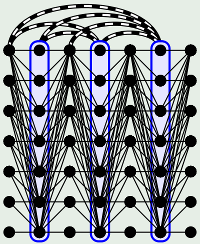
Theorem (Atminas, B., Lozin, Stacho, 2015+)

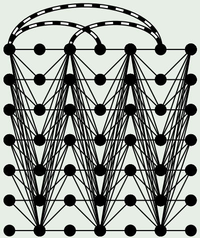
Bichain graphs are a minimal class with unbounded clique-width.

Bichain graph = union of two chains (whatever that means).

Flip edges from split permutation graphs

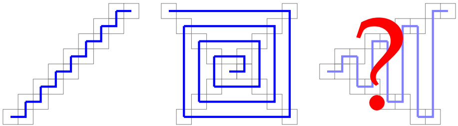
Split permutation $\rightarrow$ bichain





More minimal classes?

- Permutation class structure is a long ‘path’:



- Could find minimal classes of permutation graphs.
- Carry out edge flipping to make other graph classes.

The bad news

It looks like there are going to be lots of minimal classes with unbounded clique-width.

Linear clique-width

Set of labels Σ. You have 3 operations to build a labelled graph:

1. Create a new vertex with a label $i \in \Sigma$.
2. ~~Disjoint union of two previously-constructed graphs.~~
3. Join all vertices labelled i to all labelled j, where $i, j \in \Sigma$, $i \neq j$.
4. Relabel every vertex labelled i with j.

- Can only add vertices one at a time.
- Linear clique-width, $lcw(G) =$ size of smallest Σ to build G.

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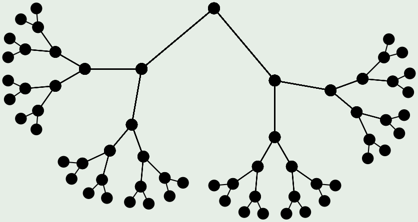
Example (Binary trees need lots of labels)



Minimal *linear* clique-width

- Clear: unbounded cw $\Longrightarrow$ unbounded lcw.
- Recent results about $\mathrm{Av}(321)$ proves the following:

Corollary (*of* Albert, B., Ruškuc, Vatter, 201?)

The class of bipartite permutation graphs is a minimal class with unbounded linear clique-width.

- Likely that all four known minimal unbounded cw classes have the same property.

Question

Do there exist classes that are minimal of unbounded clique-width, but not minimal of unbounded linear clique-width?

cw versus lcw

Question

When does a class have unbounded lcw, but bounded cw?

Two examples:

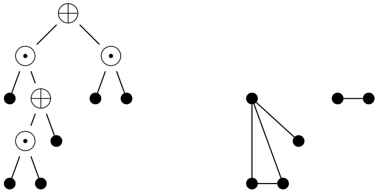
- Binary trees ($cw \leq 3$)
- Cographs ($cw = 2$): lcw is unbounded (Gurski and Wanke, 2005)

Heuristic connection

Classes which admit a tree structure of arbitrary height and width have unbounded linear clique-width.

Cographs as trees

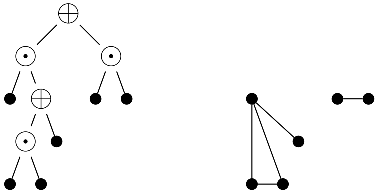
- Cographs: build from $\bullet$ by disjoint union and join
- Construct using binary trees ($\oplus$ = union, $\odot$ = join):



- Trees can be arbitrarily high and wide, so *lcw* is unbounded.

Inside cographs

- Quasi-threshold graphs: build from • by disjoint union and joining 1 new vertex



- Can use $\oplus$ freely: trees still arbitrarily high and wide, *lcw* unbounded.
- Any further restriction: width or height gets bounded. *lcw* bounded.

Inside cographs

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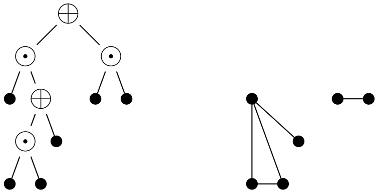


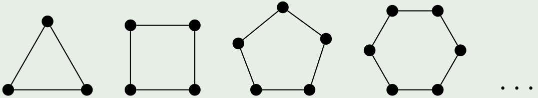
Theorem (B., Korpelainen, Vatter, 2016+)

A subclass of cographs has unbounded lcw if and only if it contains all quasi-threshold graphs, or the complement of this class.

Diversion: Infinite antichains

- Antichain: set of pairwise incomparable graphs

The set of cycles forms an antichain



Paths form a *labelled* antichain

A class is:

- well-quasi-ordered: contains no infinite antichain.
- labelled well-quasi-ordered: contains no labelled infinite antichain.

Well-quasi-order and clique-width

Conjecture (Daligault, Rao, Thomassé, 2010)

If $\mathcal{C}$ is labelled well-quasi-ordered, then $\mathcal{C}$ has bounded clique-width.

They also asked…

Question

If $\mathcal{C}$ is well-quasi-ordered, must it have bounded clique-width?

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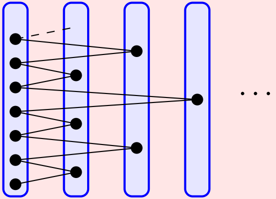
Answer is no (Lozin, Razgon, Zamaraev, 2015)



wqo, but not labelled wqo

Minimal unbounded cw and wqo

- The four known minimal unbounded clique-width classes satisfy:

Property

$\mathcal{C}$ contains a *canonical* labelled infinite antichain $\mathfrak{A}$:
If $\mathcal{D} \subset \mathcal{C}$ is a subclass with $|\mathcal{D} \cap \mathfrak{A}| < \infty$, then $\mathcal{D}$ is labelled well-quasi-ordered.

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- In each case, at most two labels are needed, so we propose:

Conjecture

Every minimal class of graphs of unbounded clique-width contains a canonical infinite antichain that uses at most two labels.

Thanks!

Main references:

- Atminas, B., Lozin & Stacho, *Minimal classes of graphs of unbounded clique-width and well-quasi-ordering*, arXiv 1503:01628.
- B., Korpelainen & Vatter, *Linear clique-width for classes of cographs*, J. Graph Theory, accepted.
- Albert, B., Ruškuc & Vatter, *Rationality for subclasses of 321-avoiding permutations*, in preparation.