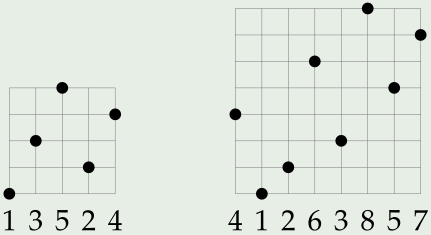
Staircases, dominoes, and the growth rate of Av(1324)

Robert Brignall

Joint work with David Bevan, Andrew Elvey Price and Jay Pantone

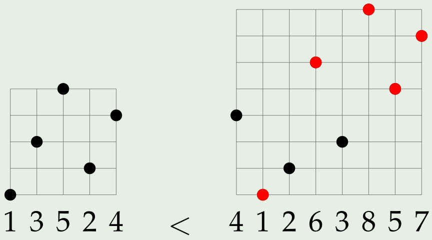
TU Wien, 28th August 2017

Permutation containment 101



- Permutations in one-line notation: $\pi = \pi(1) \cdots \pi(n)$
- Pattern containment: $\sigma \leq \pi$ if there exists a subsequence of $\pi(1) \cdots \pi(n)$ with the same relative ordering as σ.
- Containment is a partial order.
- Conversely, π avoids σ if $\sigma \not\leq \pi$.

Permutation containment 101



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Permutation containment 102

Permutation class: a hereditary collection $\mathcal{C}$, i.e.

$$\pi \in \mathcal{C} \text{ and } \sigma \leq \pi \text{ implies } \sigma \in \mathcal{C}.$$

'Principal' classes characterised by avoiding one permutation:

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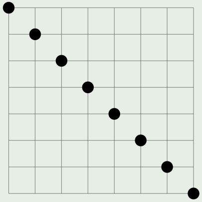
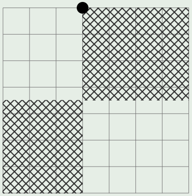


$\mathrm{Av}(12) = \{1, 21, 321, \ldots\}$ has 1 permutation of each length.



$\mathrm{Av}(132)$ has $1, 2, 5, 14, 42, \ldots$ of lengths $n = 1, 2, 3, 4, 5, \ldots$.

Counting…

...precisely

Generating function for a class $\mathcal{C}$ is the formal power series

$$f_{\mathcal{C}}(z) = \sum_{\pi \in \mathcal{C}} z^{|\pi|} = \sum_{n=1}^{\infty} |\mathcal{C}_n| z^n,$$

where $\mathcal{C}_n = \{\pi \in \mathcal{C} : |\pi| = n\}$.

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...vaguely

For principal classes $\mathrm{Av}(\beta)$, the growth rate is

$$\mathrm{gr}(\mathrm{Av}(\beta)) = \lim_{n\to\infty} \sqrt[n]{|\mathrm{Av}(\beta)_n|}.$$

Must exist due to Arratia (1999) and Marcus & Tardos (2004).

Counting Principal Classes

State of knowledge, since 1997:

β	$\lvert \mathrm{Av}(\beta)_n \rvert$	$\mathrm{gr}(\mathrm{Av}(\beta))$
1	0	0
12	1	1
123 132	$\frac{1}{n+1}\binom{2n}{n}$	4
1342 2413	$\frac{(7n^2-3n-2)}{2}(-1)^{n-1}+3\sum_{k=2}^{n}2^{k+1}\frac{(2k-4)!}{k!(k-2)!}\binom{n-k+2}{2}(-1)^{n-k}$	8
1234 1243 1432 2143	$2\sum_{k=0}^{n}\binom{2k}{k}\binom{n}{k}^2\frac{3k^2+2k+1-n-2kn}{(k+1)^2(k+2)(n-k+1)}$	9
1324	?	?

Up to symmetries, this covers all $\mathrm{Av}(\beta)$ with $\lvert\beta\rvert \le 4$.

Exact enumeration of Av(1324)

"*Not even God knows* $|Av(1324)_{1000}|$."

Doron Zeilberger, 2004

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- More recently, Conway & Guttman (2015) computed

$$|\mathrm{Av}(1324)_{36}| = 85\,626\,551\,244\,475\,524\,038\,311\,935\,717$$

Growth rate of Av(1324)

	Lower	Upper
2004: Bóna		288
2005: Bóna	9	
2006: Albert et al.	9.47	
2012: Claesson, Jelínek & Steingrímsson†		16
2014: Bóna		13.93
2015: Bóna		13.74
2015: Bevan	9.81	

2015: Conway & Guttmann estimate $\mathrm{gr}(\mathrm{Av}(1324)) \approx 11.60 \pm 0.01$

†An upper bound of 13.002 follows from an unproven conjecture.

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This work	**10.27**	**13.5**

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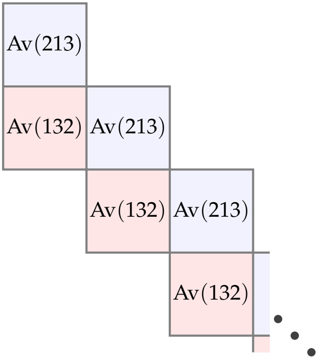
The staircase

The infinite decreasing $(\mathrm{Av}(213), \mathrm{Av}(132))$ staircase:

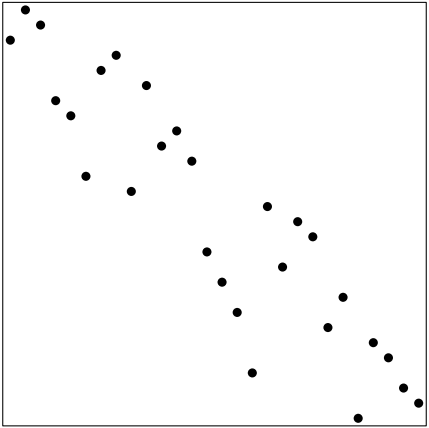
$\mathcal{S} =$



Proposition

$$Av(1324) \subset \mathcal{S}$$

Gridding a 1324-avoider



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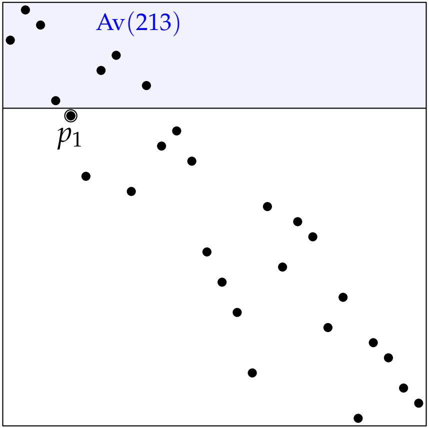
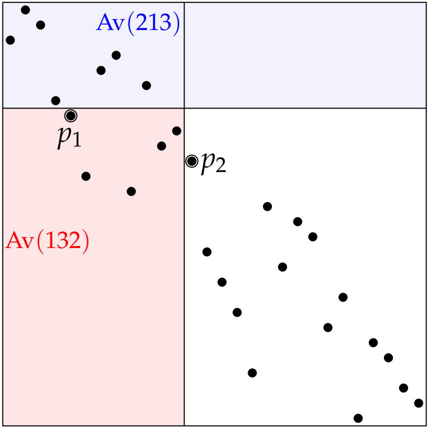


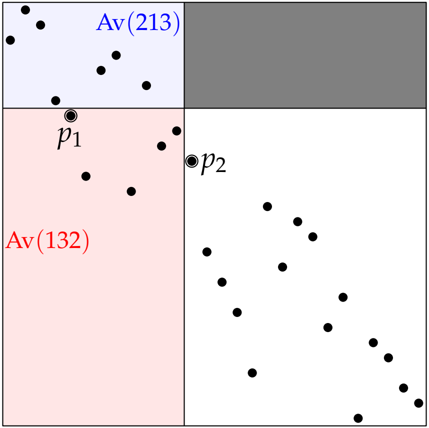
- p_1 uppermost 1 in a 213

Gridding a 1324-avoider



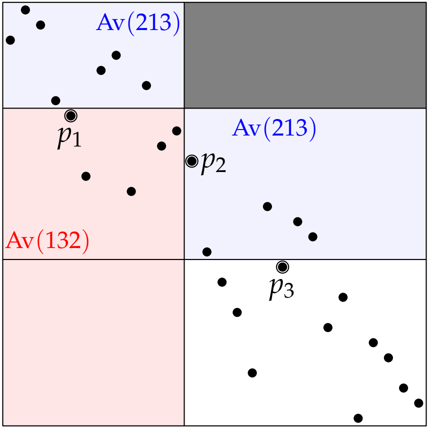
- p_1 uppermost 1 in a 213
- p_2 leftmost 2 in a 132 consisting of points below p_1 divider

Gridding a 1324-avoider



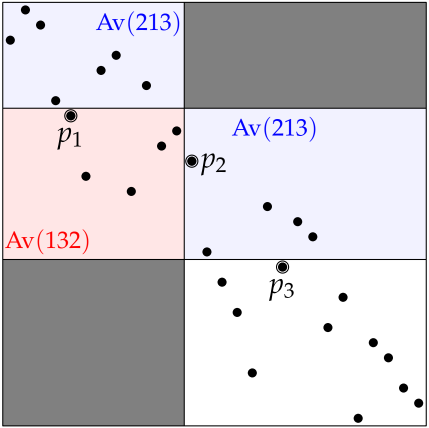
- p_2 leftmost 2 in a 132 consisting of points below p_1 divider
- No points above p_1 and to the right of p_2

Gridding a 1324-avoider



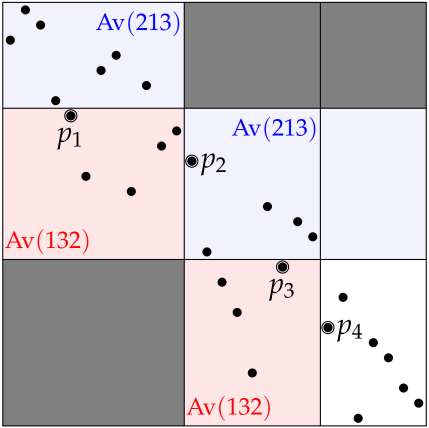
- p_2 leftmost 2 in a 132 consisting of points below p_1 divider
- p_3 uppermost 1 in a 213 consisting of points to right of p_2 divider

Gridding a 1324-avoider



- p_3 uppermost 1 in a 213 consisting of points to right of p_2 divider
- No points to the left of p_2 and below p_3

Gridding a 1324-avoider



- p_3 uppermost 1 in a 213 consisting of points to right of p_2 divider
- p_4 leftmost 2 in a 132 consisting of points below p_3 divider

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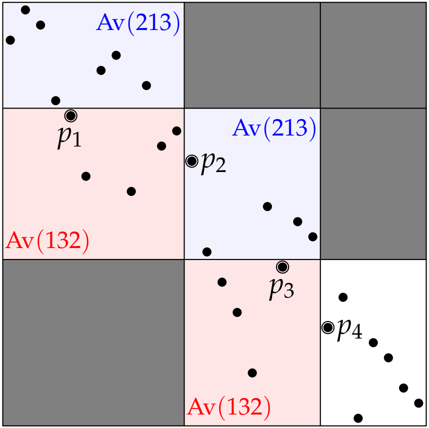
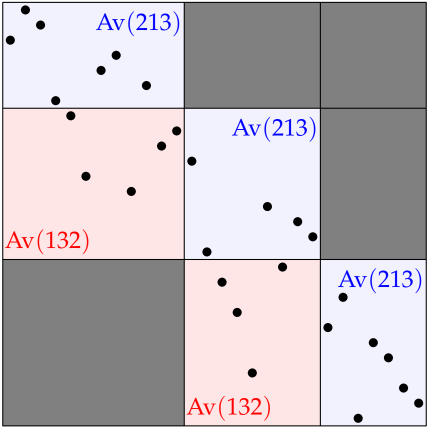


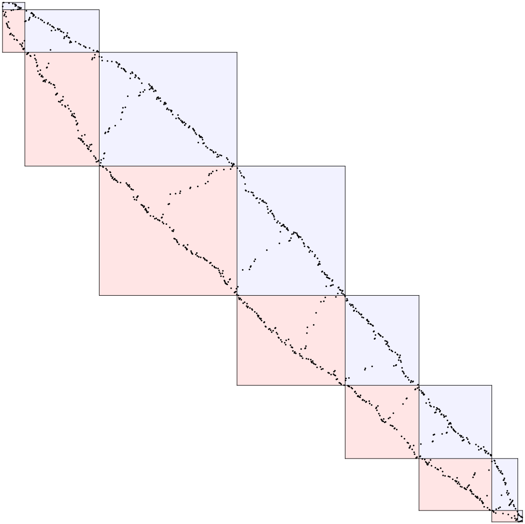
- p_4 leftmost 2 in a 132 consisting of points below p_3 divider
- No points above p_3 and to the right of p_4

Gridding a 1324-avoider



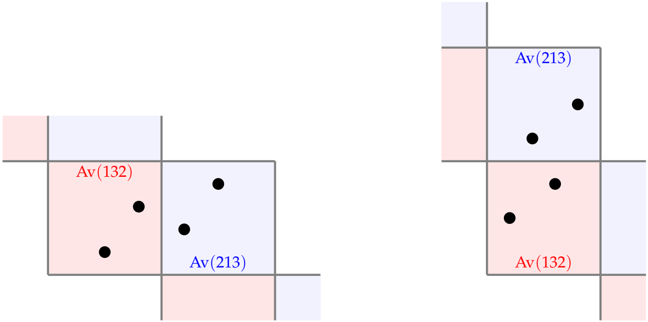
- Terminates after no more than $n/2$ steps.

The greedy gridding of a large 1324-avoider



Data provided by Einar Steingrímsson.

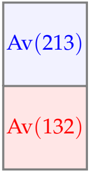
Where do I find 1324 in a staircase?



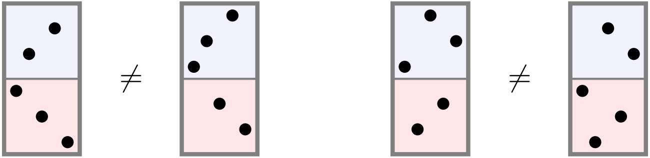
Only in two adjacent cells, and only with two points in each cell.

Dominoes

A domino is a *gridded permutation* in



that avoids 1324.



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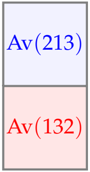

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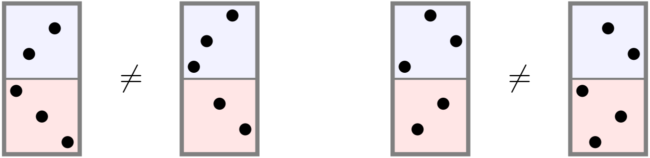
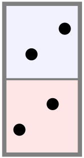


Important:  $\notin \mathcal{D}$ ($\mathcal{D}$ = the set of dominoes)

Dominoes

Theorem

The number of n-point dominoes is $\dfrac{2(3n+3)!}{(n+2)!(2n+3)!}$. $\boxed{gr(\mathcal{D}) = 27/4.}$

A familiar sequence

Among other things, dominoes are equinumerous to

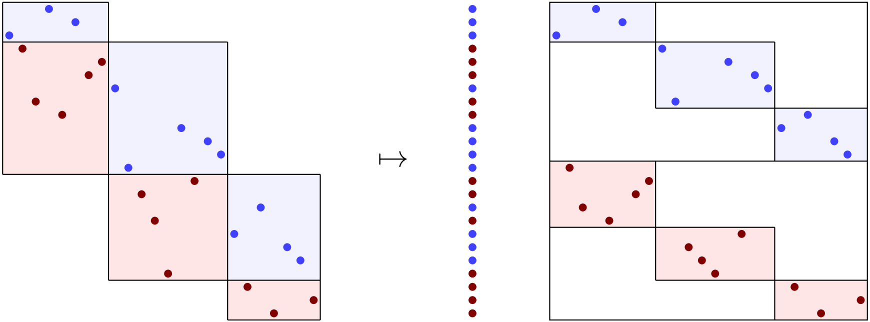
- West-2-stack-sortable permutations
- Rooted nonseparable planar maps

Problem

Find a bijection between dominoes and another combinatorial structure.

New upper bound for Av(1324)

$$\Psi : \mathrm{Av}_n(1324) \to (\textcolor{blue}{\bullet}, \textcolor{red}{\bullet})^n \times \mathcal{D}_n$$



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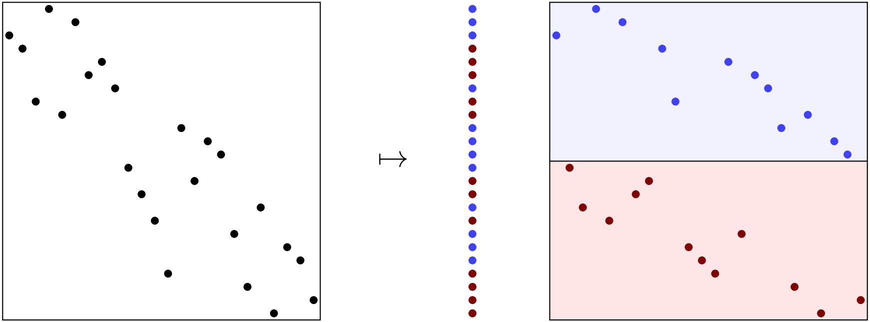

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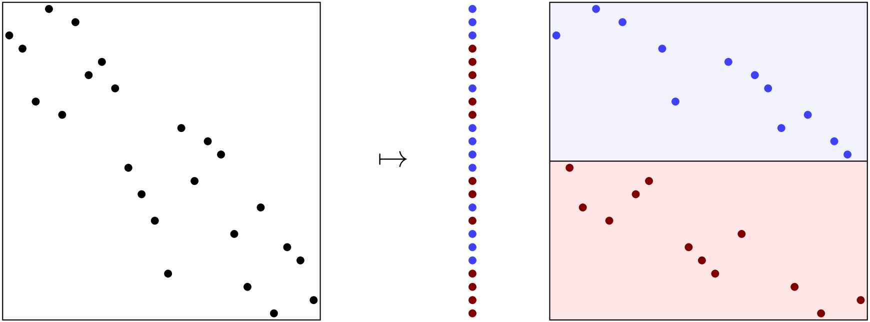
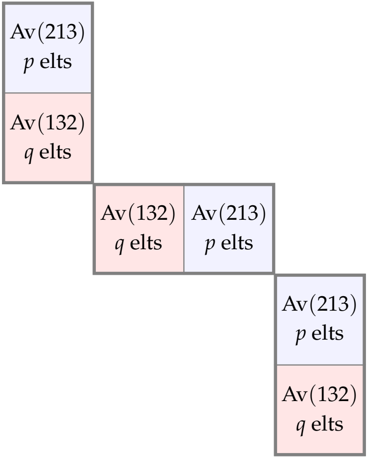


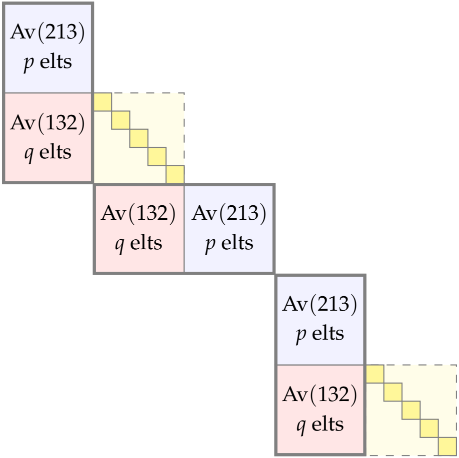
- Ψ is an injection. $\boxed{\mathrm{gr}(\mathrm{Av}(1324)) \leqslant 2 \times 27/4 = 13.5}$

New lower bound for Av(1324)



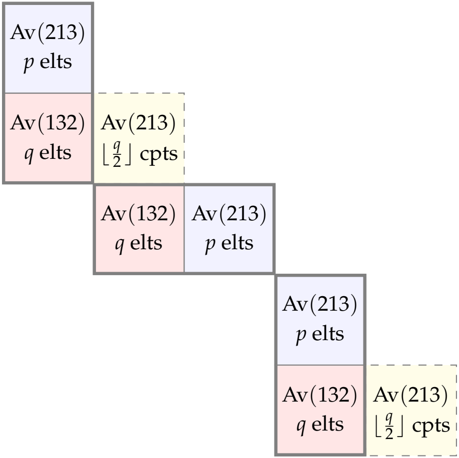
- Begin with some dominoes and their symmetries.

New lower bound for Av(1324)



- Interleave with skew indecomposable components.

New lower bound for Av(1324)



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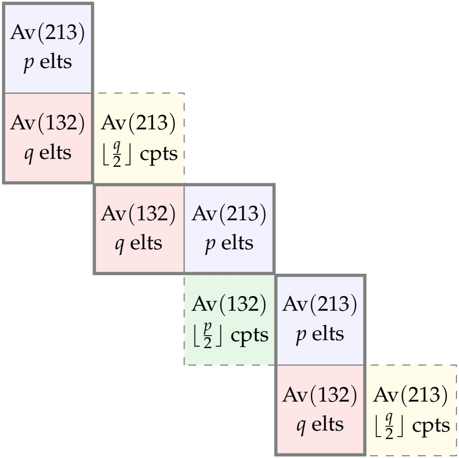

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New lower bound for Av(1324)

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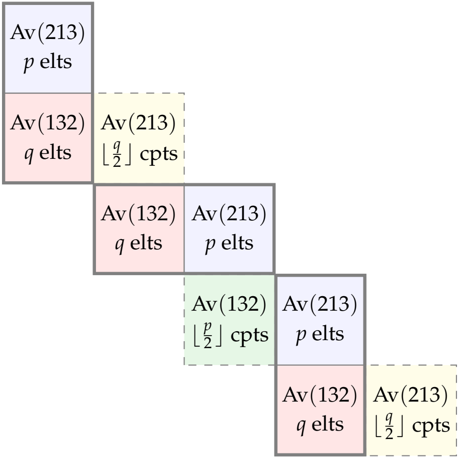
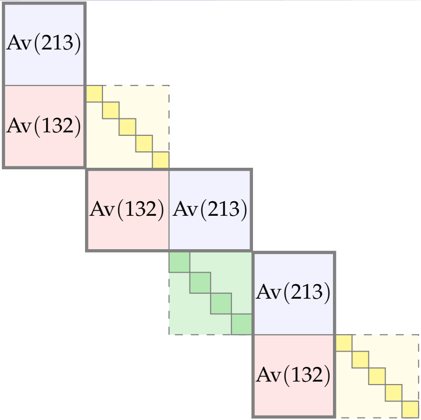


- Analysis gives $\mathrm{gr}(\mathrm{Av}(1324)) \geq 10.125$.

New lower bound (2)



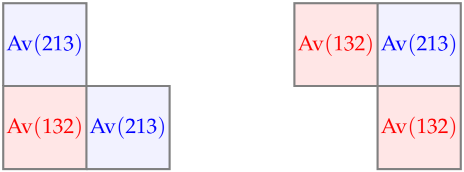
- Exploit finer structure of dominoes:

$$\boxed{\mathrm{gr}(\mathrm{Av}(1324)) \geqslant 10.2710\ldots}$$

- This is the root of a polynomial of degree 104, whose smallest coefficient has 86 digits.

Questions for the future

- Bijection between dominoes and something else
- Improvement on the (crude) upper bound
- Count trominoes?



Thanks!