Infinite Antichains in Permutation Classes

Robert Brignall

The Open University

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Outline

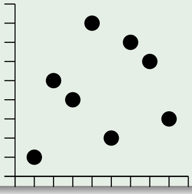
1. Introduction
   - Permutation classes
   - Enumeration
   - Antichains
2. Building antichains
   - Grid classes
   - Monotone grids
   - General grids
3. Theory of antichains
   - Intuitive structure
   - Grid pin sequences
   - Evidence for niceness

Outline

Setting the Scene

- Permutation of length n: an ordering on the symbols $1, \ldots, n$.
- For example: $\pi = 15482763$.
- Graphical viewpoint: plot the points $(i, \pi(i))$.

Example



Stack Sorting

- Knuth (1969): what permutations can be sorted through a stack?

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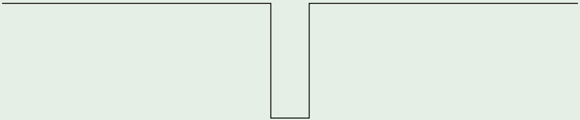

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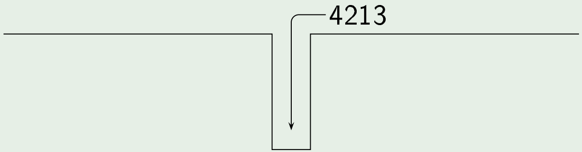

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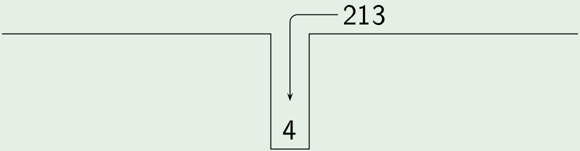

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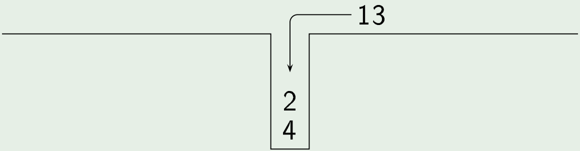

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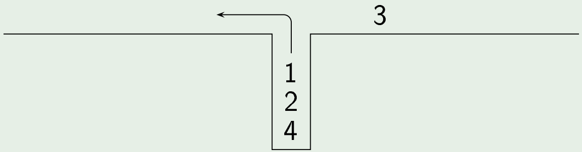

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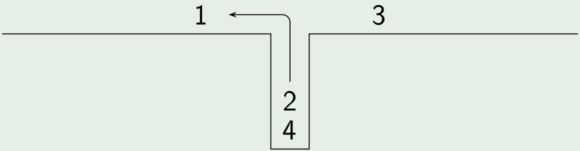

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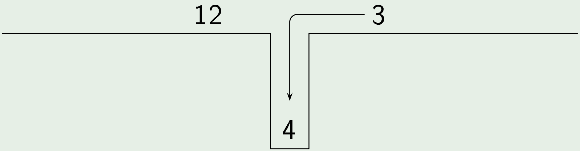

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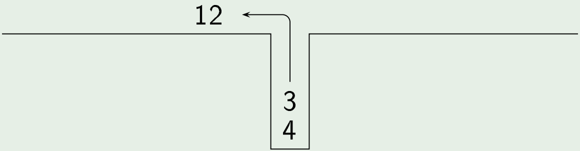

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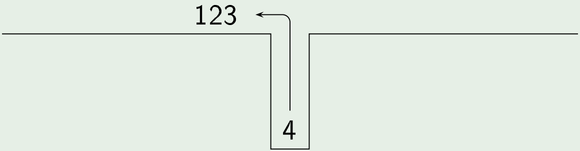

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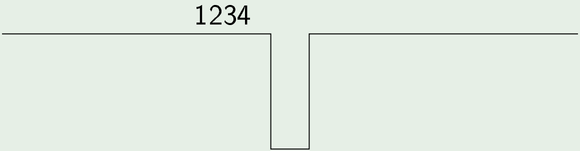

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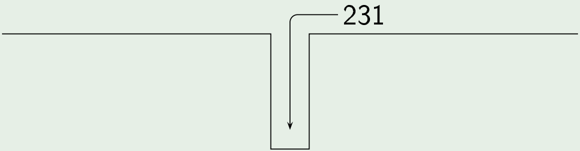

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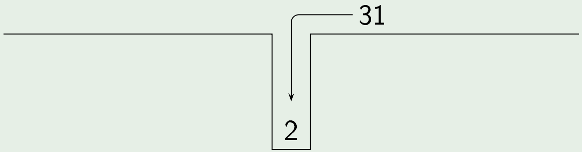

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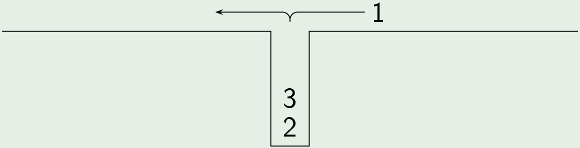

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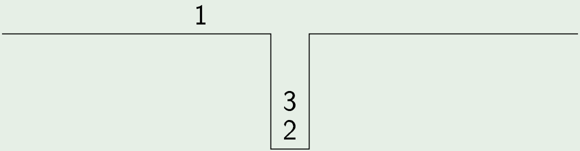

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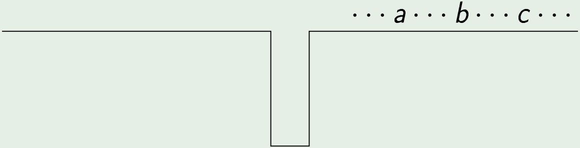
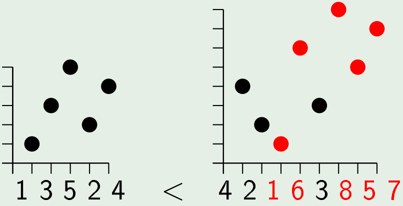


- 231 is not stack-sortable.
- In general: can't sort any permutation with a subsequence *abc* such that $c < a < b$. (*abc* forms a 231 "pattern".)

Containment

- A permutation $\tau = \tau(1) \cdots \tau(k)$ is contained in the permutation $\sigma = \sigma(1)\sigma(2) \cdots \sigma(n)$ if there exists a subsequence $\sigma(i_1)\sigma(i_2) \cdots \sigma(i_k)$ order isomorphic to τ.

Example



Permutation Classes

- Containment forms a partial order on the set of all permutations.
- Downwards-closed sets in this partial order form permutation classes. i.e. $\pi \in \mathcal{C}$ and $\sigma \leq \pi$ implies $\sigma \in \mathcal{C}$.

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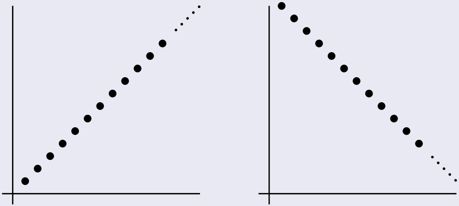
- A permutation class $\mathcal{C}$ can be seen to avoid certain permutations. Write $\mathcal{C} = \mathrm{Av}(B) = \{\pi : \beta \not\leq \pi \text{ for all } \beta \in B\}$.
- The minimal avoidance set is the basis. It is unique but need not be finite.
- E.g. the stack-sortable permutations are $\mathrm{Av}(231)$.
- Graph theoretic analogue: hereditary properties of graphs (e.g. triangle-free graphs, planar graphs, ...).

Easy Examples

- $\text{Av}(21) = \{1, 12, 123, 1234, \ldots\}$, the increasing permutations.
- $\text{Av}(12) = \{1, 21, 321, 4321, \ldots\}$, the decreasing permutations.

Typical Elements



- $\oplus 21 = \mathrm{Av}(321, 312, 231) = \{1, 12, 21, 123, 132, 213, \ldots\}$.
- $\ominus 12 = \mathrm{Av}(123, 213, 132) = \{1, 12, 21, 231, 312, 321, \ldots\}$.

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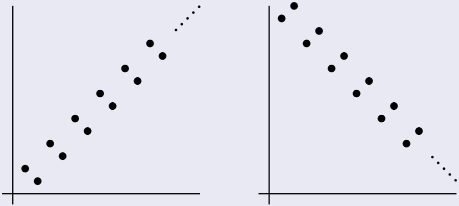


Exact Enumeration

- $\mathcal{C}_n$ – permutations in $\mathcal{C}$ of length n.
- $\sum |\mathcal{C}_n| x^n$ is the generating function.

Example

The generating function of $\mathcal{C} = \text{Av}(12)$ is:

$$1 + x + x^2 + x^3 + \cdots \quad = \quad \frac{1}{1-x}$$

Av(321) vs Av(231)

- Stack sortable permutations Av(231) enumerated by the Catalan numbers. Generating function:

$$f(x) = \frac{1 - \sqrt{1 - 4x}}{2x} = 1 + x + 2x^2 + 5x^3 + 14x^4 + \dots$$

- Using the Robinson-Schensted-Knuth correspondence with Young Tableaux, $|\mathrm{Av}(321)|_n = |\mathrm{Av}(231)|_n$.
- Despite being equinumerous, these two classes are very different: Av(321) contains infinite antichains and hence has uncountably many subclasses, while Av(231) does not.

Asymptotic Enumeration

- $\mathcal{C}_n$ – permutations in $\mathcal{C}$ of length n.

Theorem (Marcus and Tardos, 2004)

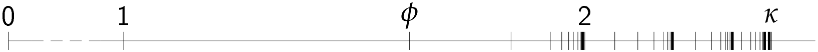
For every permutation class $\mathcal{C}$ other than the class of all permutations, there exists a constant K such that

$$\limsup_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|} \leq K.$$

- Big open question: does the growth rate, $\lim_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}$, always exist?

Small Growth Rates

- Growth rate of $\mathcal{C}$ is $\lim_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}$ (if it exists).
- Below $\kappa \approx 2.20557$, growth rates exist and can be characterised [Vatter, 2007+]:

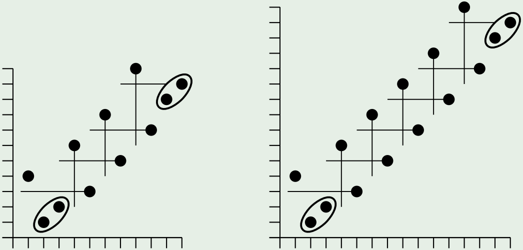


- κ is the lowest growth rate where we encounter infinite antichains, and hence uncountably many permutation classes.
- The proof of this uses grid classes (more on this later).

Infinite Antichains

- (Infinite) set of pairwise incomparable permutations.

Example (Increasing Oscillating Antichain)



- N.B. These permutations avoid 321.

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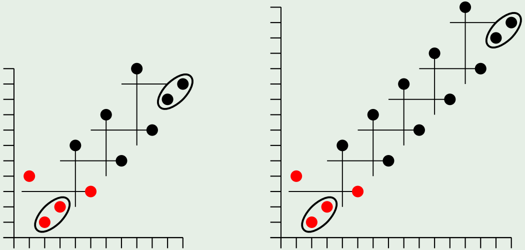

- Anchor: bottom copies of 4123 must match up.

- (Infinite) set of pairwise incomparable permutations.

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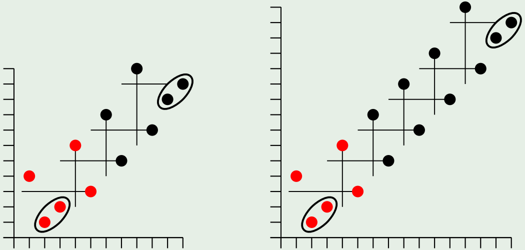

- Each point is matched in turn.

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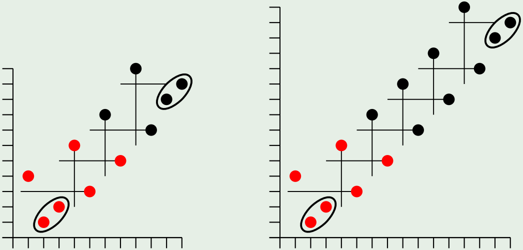

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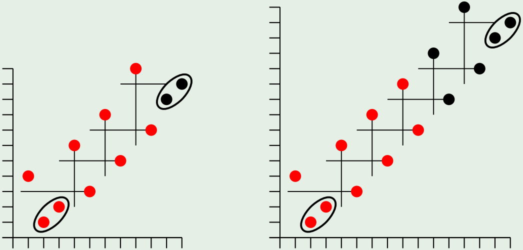

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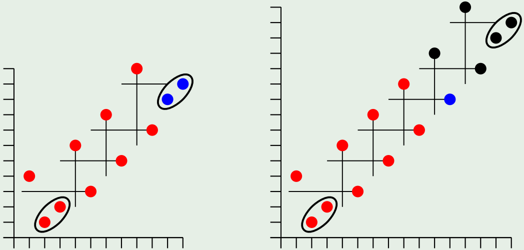
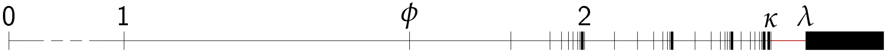


- Last pair cannot be embedded.

Increasing Oscillations are Important

- At $\kappa \approx 2.20557$, we find permutation classes that contain the increasing oscillating antichain.
- Above $\lambda \approx 2.48188$, every real number is the growth rate of a permutation class [Vatter, 2010].
  The proof builds classes based on this antichain.



- From order to chaos: What lies between κ and λ?

When are there antichains?

No infinite antichains.

- Words over a finite alphabet [Higman, 1952].
- Trees ordered by topological minors [Kruskal 1960; Nash-Williams, 1963]
- Graphs closed under minors [Robertson and Seymour, 1983—2004].

Infinite antichains.

- Graphs closed under induced subgraphs (or merely subgraphs). e.g. $C_3, C_4, C_5, \ldots$
- Permutations closed under containment.
- Tournaments, digraphs, ...

Partial Well-Order

- There exist infinite antichains in the permutation poset, but not every class has then.
- A permutation class is partially well-ordered (pwo) if it contains no infinite antichains.

Question

Can we decide whether a permutation class given by a finite basis is pwo?

- To prove pwo — Higman's theorem is useful.
- To prove not pwo — find an antichain.

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Question

Can we decide whether a hereditary property given by a finite basis is wqo?

- To prove pwo — Higman's theorem is useful.
- To prove not pwo — find an antichain.
- Other structures: well quasi-order, not pwo, but same idea.

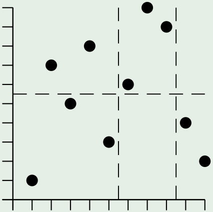
Outline

Grid Classes

- Hot topic: Crucial tool to study the structure of classes.
- Matrix $\mathcal{M}$ whose entries are (infinite) permutation classes.
- $\mathrm{Grid}(\mathcal{M})$ the grid class of $\mathcal{M}$: all permutations which can be "gridded" so each cell satisfies constraints of $\mathcal{M}$.

Example

- Let $\mathcal{M} = \begin{pmatrix} \mathsf{Av}(21) & \mathsf{Av}(231) & \emptyset \\ \mathsf{Av}(123) & \emptyset & \mathsf{Av}(12) \end{pmatrix}$.



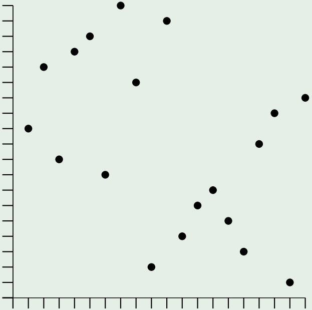
$\in \mathrm{Grid}(\mathcal{M})$

Monotone Grid Classes

- Special case: all cells of $\mathcal{M}$ are Av(21) or Av(12).
- Rewrite $\mathcal{M}$ as a matrix with entries in $\{0, 1, -1\}$.

Example

$$\mathcal{M} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$



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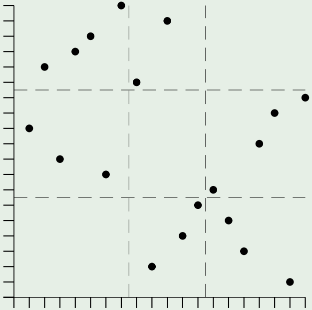

The Graph of a Matrix

- Graph of a matrix, $G(\mathcal{M})$, formed by connecting together all non-zero entries that share a row or column and are not "separated" by any other nonzero entry.

Example

$$\begin{pmatrix} \mathcal{C} & 0 & 0 & \mathcal{D} \\ 0 & 0 & \mathcal{E} & 0 \\ \mathcal{D} & \mathcal{E} & 0 & \mathcal{C} \\ 0 & 0 & 0 & \mathcal{D} \end{pmatrix}$$

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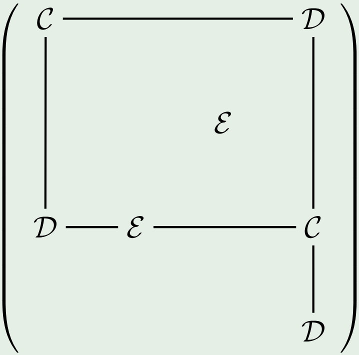


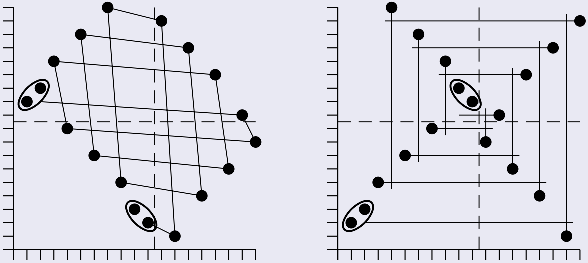
Monotone Grids and Partial Well-Order

Theorem (Murphy and Vatter, 2003)

The monotone grid class $\mathrm{Grid}(\mathcal{M})$ *is pwo if and only if* $G(\mathcal{M})$ *is a forest, i.e.* $G(\mathcal{M})$ *contains no cycles.*

Proof.

($\Rightarrow$) Construct infinite antichains that "walk" around a cycle.



□

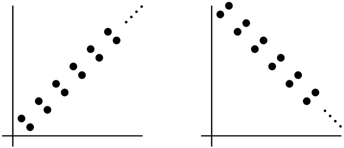
When does that apply?

Question

When is a class $\mathcal{C}$ (a subset of) a monotone grid class?

Answer [Huczynska & Vatter]

A class $\mathcal{C}$ is monotone griddable if and only if it contains neither the classes $\oplus 21$ nor $\ominus 12$.



Beyond monotone

- What can we say about infinite antichains for general grid classes?
- Next stage: allow cells labelled by $\oplus 21$ and $\ominus 12$.

Example

$$\begin{pmatrix} \mathrm{Av}(21) & 0 & 0 & \mathrm{Av}(21) \\ 0 & \ominus 12 & 0 & 0 \\ \oplus 21 & 0 & \mathrm{Av}(12) & 0 \\ 0 & 0 & 0 & \oplus 21 \end{pmatrix}$$

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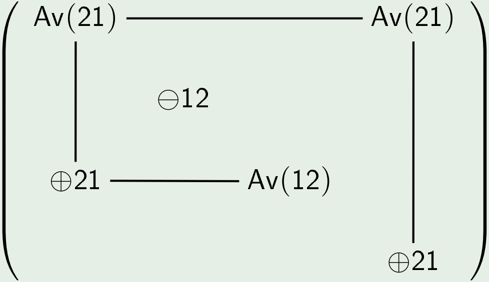


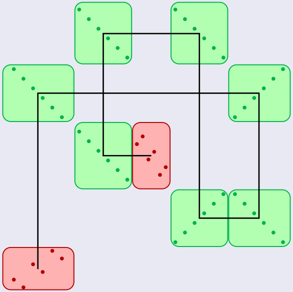
- Can assume graph is a forest, but now the number of non-monotone-griddable cells in each component matters.

Two is too many

Theorem (B.)

A grid class whose graph has a component containing two or more non-monotone-griddable cells is not pwo.

Proof.



- WLOG graph is a path connecting two bad cells.

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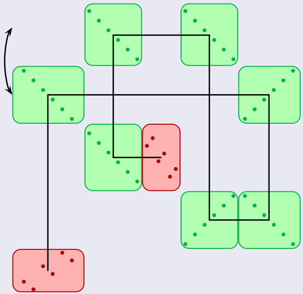

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- Permute rows and columns.

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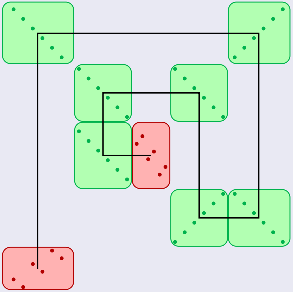

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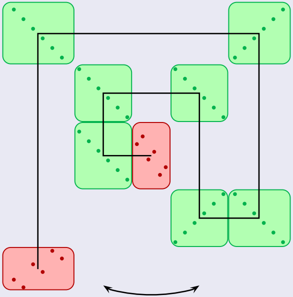

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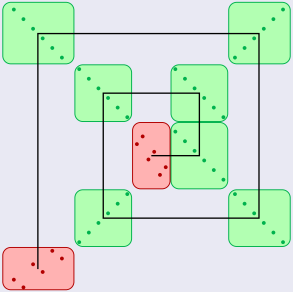

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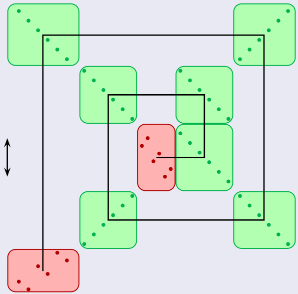

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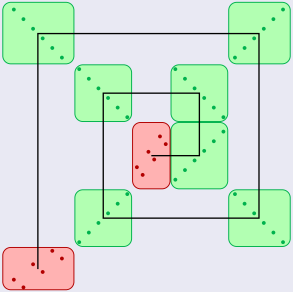

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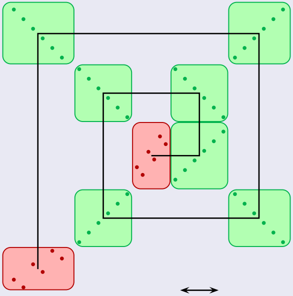

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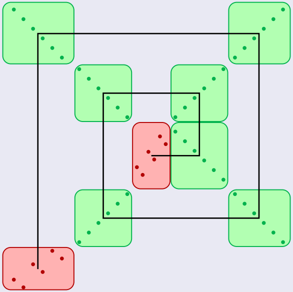

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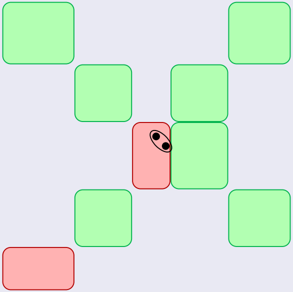

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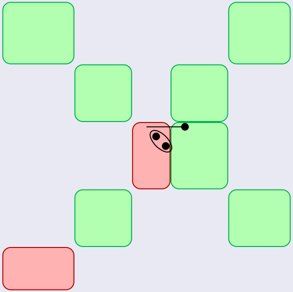

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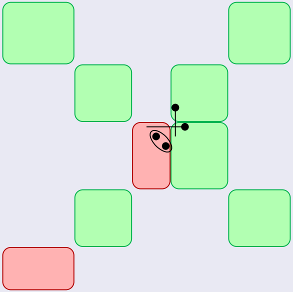

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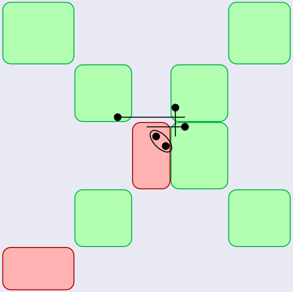

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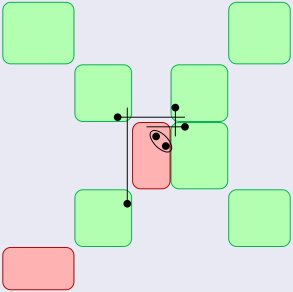

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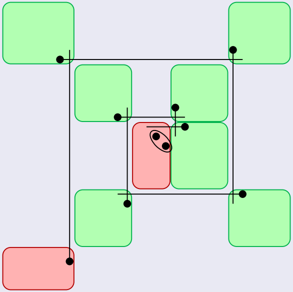

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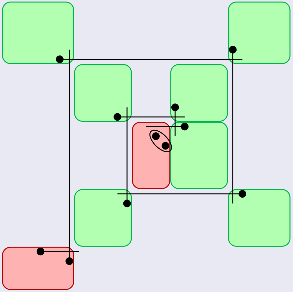

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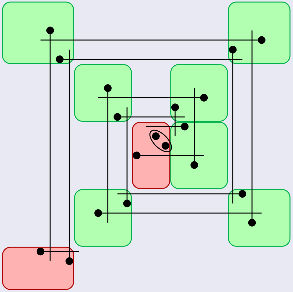

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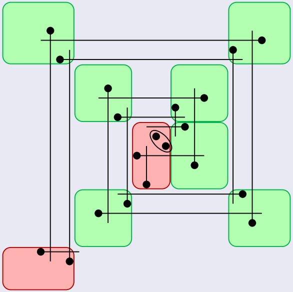

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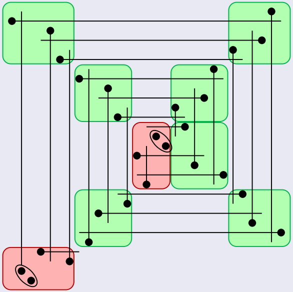

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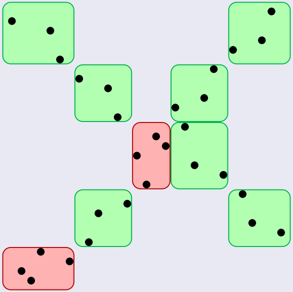

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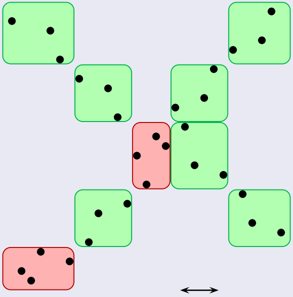

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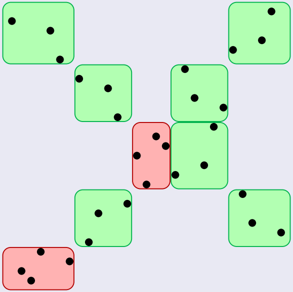

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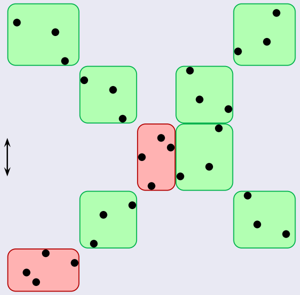

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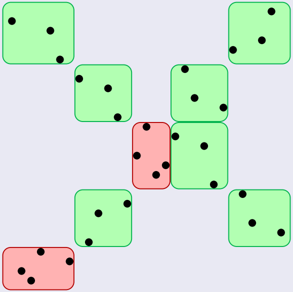

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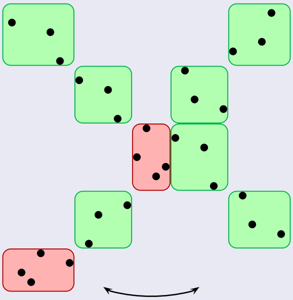

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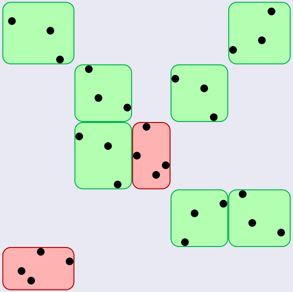

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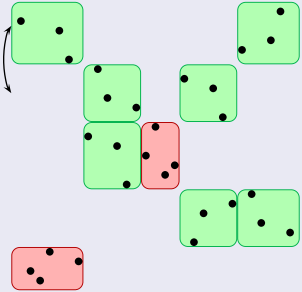

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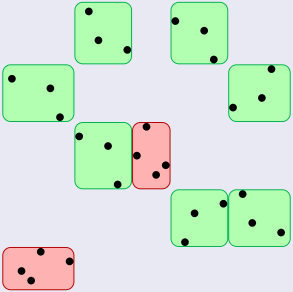
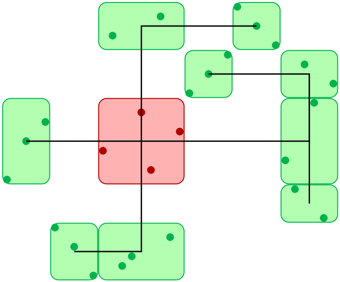


- WLOG graph is a path connecting two bad cells.
- Permute rows and columns.
- Flip rows and columns.
- Build antichain with grid pin sequences.
- Flip and permute back.
- Still have an antichain.

□

Just one non-monotone

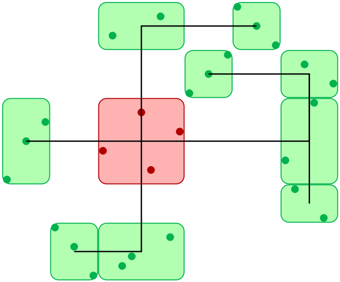
- What if a component contains exactly one non-monotone griddable cell?
- First: Add the (fairly strong) condition that the “bad” cell contains only finitely many simple permutations.
- Now can describe the class in a way which is amenable to Higman's Theorem.



Just one non-monotone

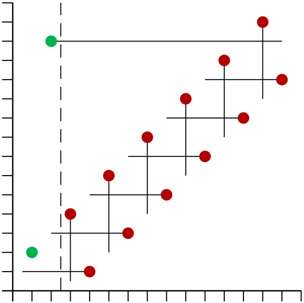
Theorem (B.)

Let $\mathcal{M}$ be a gridding matrix for which each component is a forest and contains at most one non-monotone cell. If every non-monotone cell contains only finitely many simple permutations, then $\mathrm{Grid}(\mathcal{M})$ *is pwo.*



But sometimes one is too much...

- One cell containing arbitrarily long increasing oscillations next to a monotone cell is bad...



- Mind the gap: between finite simples and infinite oscillations, not (yet) known.

Outline

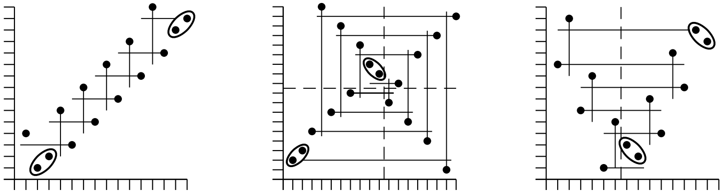
1. Introduction
   - Permutation classes
   - Enumeration
   - Antichains
2. Building antichains
   - Grid classes
   - Monotone grids
   - General grids
3. Theory of antichains
   - Intuitive structure
   - Grid pin sequences
   - Evidence for niceness

The way forward. . .

- Grand aim: a structure theory for infinite antichains, to answer (or explain why we can't answer) questions about partial well-order.
- In this talk: restrict attention to permutations, but this theory is really for general combinatorial structures.

Intuitive structure of antichains

- Take an infinite sequence of points in the plane, $p_1, p_2, \ldots,$ each following on "uniquely" from its predecessors.
- Antichain elements: take a finite sequence $p_1, \ldots, p_n$ of these points, and blow up the first and last points.
- Alternative to blowing up: tie the ends together.



Question

Is this intuitive description of structure correct?

Caveats

Antichains can be more complicated, but we don't care:

- An infinite antichain A is fundamental if its closure,

$$\mathrm{Cl}(A) = \{\pi : \pi \leq \alpha \text{ for some } \alpha \in A\},$$

  contains no infinite antichains other then subsets of A.
- Fundamental really means no extraneous points.
- Related concepts: minimal, maximal, canonical...

Proposition (Essentially due to Nash-Williams, 1963)

Every non-pwo permutation class contains a fundamental infinite antichain.

Bigger caveat: Maybe we just haven't found any ugly antichains yet.

Grid pin sequences

- Local separation: p_{i+1} separates p_i from p_{i-1}.
- Local externality: p_{i+1} lies outside $\mathrm{Rect}(p_{j-1}, p_j)$, $j = 1, \ldots, i$.
- Row-column agreement: p_{i+1} must be placed in the same row or column as p_i.
- Non-interaction: p_{i+1} could not have been used as a pin earlier in the sequence.

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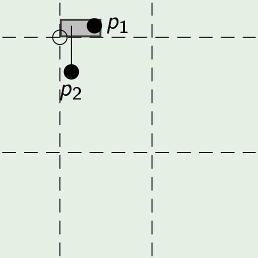

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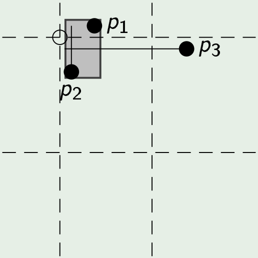

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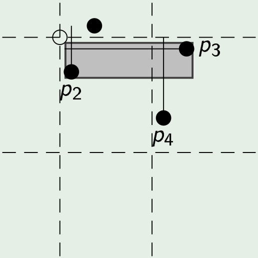

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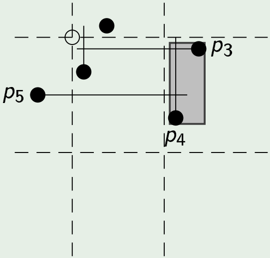

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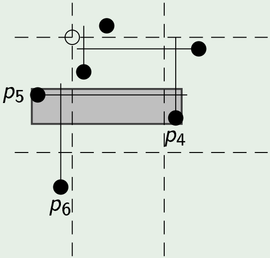

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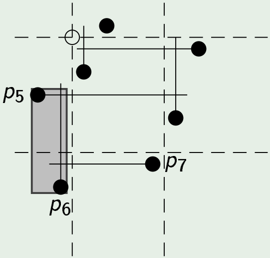

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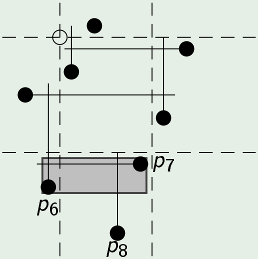

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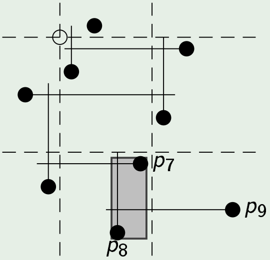

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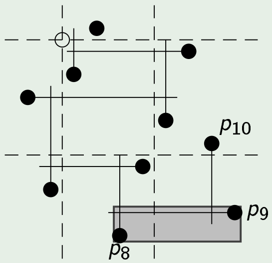


To other structures

- Grid pin sequences transfer to other combinatorial structures.
- Translation resolves a conjecture in graph theory:

Conjecture (Ding, 1992)

The class of permutation graphs that do not contain (as an induced subgraph) a path or the complement of a path on $n \geq 5$ vertices is wqo.

Counterexample

Permutations	$\rightarrow$	Permutation graphs
Increasing oscillations (no blow-up)	$\rightarrow$	Paths
Decreasing oscillations	$\rightarrow$	Complement of paths

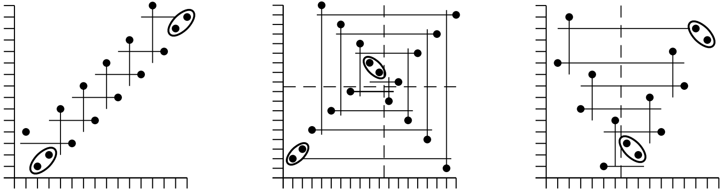
The "Widdershins" antichain (see next slide) lies in this class.

Evidence of niceness

Theorem (Cherlin and Latka, 2000)

For each natural number k, there is a finite set Λ_k of fundamental antichains with the property that a class avoiding exactly k permutations is pwo if and only if its intersection with each antichain in Λ_k is finite.

- Λ_1 consists of the increasing oscillating, Widdershins and V antichains [Atkinson, Murphy and Ruškuc, 2002].

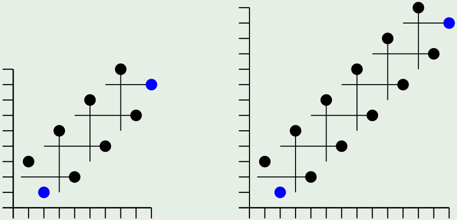


- Λ_2 is unknown...

Colour your permutations

- Permutations with $n \geq 2$ colours: no blow up required.
- n-pwo: permutation class contains no n-coloured infinite antichains.

Example (2-Coloured Increasing Oscillating Antichain)

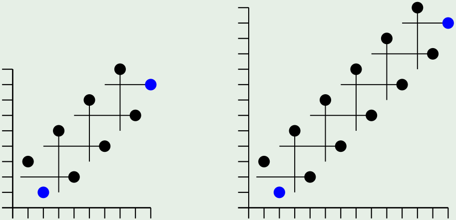


Conjecture (Pouzet, 1972)

A permutation class $\mathcal{C}$ is 2-pwo if and only if $\mathcal{C}$ is n-pwo for all $n \geq 2$.

(N.B. This is really a conjecture about graphs.)

Example (2-Coloured Increasing Oscillating Antichain)



Questions and Summary

- Conjectures describing the "niceness" of antichains are abundant. Proofs are scarcer.
- Permutations: what does antichain structure mean for permutation class structure?
- Could a better understanding of infinite fundamental antichains fill the gap between existing antichain constructions and techniques for proving pwo?

Thanks!